Adaptive Time Domain Boundary Element Method for Sound Radiation of Tyres

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Abstract

This thesis discusses the numerical solution of time-dependent scattering phenomena in the half-space using retarded potential boundary integral equations. We consider the case of a sound-hard, sound-soft or partially absorbing surface with a Robin boundary condition. Time-domain boundary integral operators adapted to the surface are used to solve this boundary problem. For the analysis of the boundary integral equations we follow the approach of Bamberger and Ha-Duong and consider first the corresponding boundary value problem for the Helmholtz equation in the frequency domain. We show that in physically relevant situations the problems are well-posed and, in particular, uniquely solvable.

For the bilinear form belonging to the Calderon projector we show coercivity and continuity.

We then discuss the efficient numerical solution of the boundary problem based on a time-domain Galerkin boundary element method in $\mathbb{R}^3_+$. A priori and a posteriori error estimates are presented in space-time Sobolev norms. In particular, we show the reliability of a residual type a posteriori error estimate for both the integral equation of the first kind with the single layer potential and for the complete system associated to the Calderon projector for the Robin problem with absorbing boundary conditions in $\mathbb{R}^3_+$. The a priori estimates guarantee the convergence of our methods for the Dirichlet and Robin problems in the half-space.

Numerical experiments for a spherical obstacle in the half-space confirm our theoretical predictions in a simple situation, where an exact solution is available. We use the a posteriori error estimates to define an adaptive time-domain boundary element procedure for singular boundary data and thereby make a first step towards space-time adaptive methods.

Numerical experiments on the sound radiation of real-world tyre models show the applicability of the developed methods to engineering problems. We compare our time-domain simulations with the results from stabilized frequency-domain boundary element methods. The experiments concern the sound radiation for given tyre vibrations, the amplification of sound in the horn-like geometry where the tyre meets the street as well as the Doppler shift of sound frequencies for a moving tyre.

Keywords. boundary element method, retarded potentials, absorbing half-space, a proiri, a posteriori error estimates, tyre.

Zusammenfassung

Diese Dissertation betrachtet die numerische Lösung zeitabhängiger Streuprobleme im Halbraum mit Hilfe zeitabhängiger Randintegralgleichungen. Wir betrachten schallharte, schallweiche oder teilweise absorbierende Oberflächen mit einer Robin-Randbedingung. Zur Lösung des betrachteten Randwertproblems erweisen sich an die Oberfläche angepasste Randintegraloperatoren als hilfreich.
Die Analyse der sich ergebenden Integralgleichungen folgt einem auf Bamberger und Ha-Duong zurückgehenden Zugang. Dafür betrachten wir zuerst ein Fourier-transformiertes Randwertproblem für die Helmholtzgleichung im Frequenzbereich. Wir zeigen, dass dieses in den physikalisch relevanten Fällen wohlgestellt und insbesondere eindeutig lösbar ist. Darüber hinaus weisen wir für eine mit Hilfe des Calderon-Projektors definierte Bilinearform die Koerzivität und Stetigkeit nach. Für die effiziente numerische Lösung des Randwertproblems präsentieren wir eine zeitabhängige Galerkin Randelementmethode im $\mathbb{R}^3_+$. A priori and a posteriori Fehlerabschätzungen in Raum-Zeit-Sobolevnormen werden hergeleitet. Insbesondere beweisen wir die Zuverlässigkeit eines residualen a posteriori Fehlerschätzers sowohl für die Integralgleichung erster Art mit dem Einfachschichtpotential als auch für das vollständige zum Calderon-Projektor gehörende System von Gleichungen für das absorbierende Robin-Problem im $\mathbb{R}^3_+$.
Numerische Experimente für eine Kugel als Hindernis im Halbraum bestätigen unsere theoretischen Untersuchungen in einem einfachen Fall, in dem wir eine exakte Lösung angeben können. Basierend auf den a posteriori Abschätzungen definieren wir eine adaptive, zeitabhängige Randelementmethode für singuläre Randdaten. Diese stellt einen ersten Schritt hin zu allgemeineren Raum-Zeit-adaptiven Randelementmethoden dar.
Numerische Experimente zur Schallabstrahlung realistischer Reifenmodelle zeigen die Anwendbarkeit der entwickelten Methoden auf Ingenieurprobleme. Wir vergleichen unsere Simulationen im Zeitbereich mit Ergebnissen stabiliserter frequenzabhängiger Randelementmethoden. Die Experimente betrachten die Schallabstrahlung bei vorgegebenen Reifenschwingungen, die Verstärkung des Schalls in der Horngeometrie zwischen Reifen und Straße sowie die Dopplerverschiebung der Schallfrequenz für einen bewegten Reifen.

Schlagworte. Randelementemethode, retardierte Potentiale, absorbierender Halbraum, a priori, a posteriori Fehlerabschätzungen, Reifen.

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0 Introduction

Many important physical applications such as electromagnetic wave propagation or the computation of transient acoustic waves are governed by the wave equation. Such problems are posed in unbounded domains and the method of integral equations is used to transform the wave equation in the integral equation on the boundary surface of the scatterer. Numerical discretization methods are collocation methods with stabilization techniques [16] [17] [18], Laplace-Fourier methods coupled with Galerkin boundary elements in space [4] [13] [22] [23] [46] [47] and the convolution quadrature method for the time discretization [7][31].
This thesis treats the time domain boundary element approach and is based on the above mentioned Laplace-Fourier methods. This approach allows to describe the time-wise behaviour of the sound radiation of car tyres, e.g. one can compute the Doppler effect for a moving car, see Chapter 6.
It is split into theoretical parts (Chapters 1-4) and numerical parts (Chapters 5-6). After the introductory Chapter 1 with the necessary definitions of the Fourier Laplace transformation, anisotropic Sobolev spaces and boundary element spaces, we introduce in Chapter 2 the retarded potentials of the single layer and of the double layer and the corresponding boundary integral operators together with the jump relations. Then we briefly describe the marching-on-in-time (MOT) scheme, which is used to solve the space time discretized exterior Dirichlet, resp. Neumann problem in $\mathbb{R}^3$ based on a Galerkin scheme for the boundary integral equations. A summary of known results for the retarded single layer potential ansatz to treat the Dirichlet problem closes Chapter 2.
In the following Chapter 3 we consider the boundary value problem for the wave equation in the exterior domain in $\mathbb{R}^3_+$ of a bounded domain Ω with Lipschitz boundary Γ. Especially the case of a partially absorbing surface $\Gamma_\infty = \partial\mathbb{R}^3_+ = \mathbb{R}^2$ with a Robin boundary condition is considered. This boundary value problem is solved with a suitable transient Green's function for the half-space. An important step here is the derivation of a representation formula for the solution (Theorem 3.1). For the analysis of the boundary integral equations we follow the approach of Bamberger and Ha-Duong and consider first the corresponding boundary value problem for the Helmholtz equation in the frequency domain. We show that in physically relevant situations the problems are well-posed and uniquely solvable. For the bilinear form a_ω in (3.17) belonging to the Calderon projector we show coercivity (Theorem 3.3) and continuity (Theorem 3.5). Then we consider in the time domain the corresponding boundary integral equation for the acoustic scattering problem with a Robin boundary condition on Γ and Γ_∞.

Like Bamberger and Ha-Duong, we can go from the time harmonic case to the time-dependent case with the help of the inverse Fourier Laplace transformation, and hence we can show continuity and coercivity of the corresponding bilinear form in (3.33) in anisotropic Sobolev spaces spaces (Theorem 3.7).
Chapter 4 decomposes into 2 parts: a) a priori error estimates in space-time Sobolev spaces for the time-domain boundary element method in $\mathbb{R}^3_+$ (Section 4.1). b) reliability of a residual type a posteriori error estimate for integral equations of the first kind with the single layer potential (Section 4.2) and the complete system of the Calderon projector for the Robin problem with absorbing boundary conditions (Section 4.3) in $\mathbb{R}^3$. The a priori error estimate in Section 4.1.2 concerns the initial boundary value problem with Robin boundary conditions on $\Gamma \cup \Gamma_\infty$. We obtain this result (Theorem 4.2) by extending the techniques of Bamberger and Ha-Duong, there only Dirichlet or Neumann problems are considered in $\mathbb{R}^3$. In our derivation of our a posteriori error estimator we extend the approach of Carstensen and Stephan (for the elliptic case) to our hyperbolic situation (Theorems 4.3 and 4.4). In Section 4.4 we present an adaptive algorithm for space-time refinement and give numerical experiments for the first kind integral equation with the single layer potential.
The second part of the dissertation starts in Section 5.1 with the description of the boundary element method for the exterior boundary problem of the wave equation in the half-space $\mathbb{R}^3_+$ with Robin boundary conditions on Γ_∞. Here we extend the approach by Ostermann (there for $\mathbb{R}^3$ and Dirichlet boundary conditions) to our situation. Then the Neumann problem in the absorbing half-space is considered which leads to the integral equation (5.8) of the second kind with the normal derivative of the single layer potential. In Section 5.2 we first present numerical experiments for the half-space case outside a sphere. Here, an exact solution is known and can be used for validation. Then in Section 5.2.2 we consider the benchmark of the sound radiation of a tyre with sound hard road Γ_∞. Again the core part is the implementation of the BEM for the second kind integral equation with the normal derivative of the single layer potential (5.21), now with a physical right hand side. The benchmark of a vibrating tyre is computed in Section 5.2.3. The horn effect, another benchmark, is considered in Section 5.2.4. Again numerical computations are compared with the engineering data.
Chapter 6 is dedicated to the analysis of a rolling tyre. First the corresponding Green's function is given and an exact solution is obtained via Lorentz transformation. Our numerical experiments in Section 6.3 show the Doppler effect which appears when the tyre moves towards the observer.
Our numerical experiments are carried out on an Intel Xeon computer server at the Institute of Applied Mathematics at LUH with an extended version of the software package maiprogs.

1 Notation and Definitions

1.1 Notation and Definitions:

We start this chapter with a brief introduction into the main concepts and definitions connected with the Sobolev spaces used and some standard notations for distributions (see e.g. [2]).
First we recall the definitions of the Fourier transform and the Laplace transform. For $u(t,.) \in \mathcal{S}(\mathbb{R})$, the Schwartz space of tempered functions, and $\eta \in \mathbb{R}$, the Fourier transform with respect to the time variables is given by

$$\mathcal{F}_t[u(t,.)](\eta,.) = \int_{\mathbb{R}} e^{i\eta t} u(t,.)dt,$$

and for $u(.,x) \in \mathcal{S}(\mathbb{R}^N)$ and $\xi \in \mathbb{R}^N$ the Fourier transform with the respect to the space variables is given by

$$\mathcal{F}_x[u(.,x)](.,\xi) = \int_{\mathbb{R}^N} e^{i\xi.x} u(.,x)dx,$$

Finally, for $\omega = \eta + i\sigma \in \mathbb{C}$ and for $u(t,.) \in LT$ (the space LT is introduced below), the Fourier-Laplace transform with respect to the time variables is given by

$$\mathcal{L}_t[u(t,.)](\omega,.) = \int_{\mathbb{R}} e^{i\omega t} u(t,.)\, dt.$$

We sometimes write $\hat{u}$ instead of $\mathcal{L}_t[u]$.
We introduce the standard definition of the $L^2(\Omega)$-space as the set of all functions $u : \Omega \longrightarrow \mathbb{R}$ which are square-integrable over Ω in the sense of Lebesgue. $L^2(\Omega)$ is a Hilbert space with the scalar product

$$(u,v)_0 = (u,v)_{L^2} = \int_{\Omega} u(x)v(x)\, dx$$

and the corresponding norm

$$||u||_0 = \sqrt{(u,u)_0}\,.$$

For $u \in L^2(\Omega)$, $\partial^\alpha u$ represents the weak derivative in $L^2(\Omega)$ which is given by

$$\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$$

where $\alpha = (\alpha_1, ..., \alpha_d)$ with $\alpha_i \in \mathbb{N}_0$ is a multiindex , $|\alpha| := \alpha_1 + \cdots + \alpha_d$. Assuming that $\partial^\alpha u \in L^2(\Omega)$ we get

$$(\varphi, \partial^\alpha u)_0 = (-1)^{|\alpha|}(\partial^\alpha \varphi, u)_0 \quad \forall \varphi \in C_0^\infty(\Omega).$$

We denote by $C^\infty(\Omega)$ the space of infinitely times differentiable functions on Ω and by $C_0^\infty(\Omega)$ the subspace of functions with compact support in Ω, i.e. functions which are non zero only on a compact subset of Ω.
We define the Sobolev space $H^m(\Omega)$ for a given integer $m \geq 0$ by

$$H^m(\Omega) = \{u \in L^2(\Omega)| \quad \partial^\alpha u \in L^2(\Omega) \quad \forall |\alpha| \leq m\}.$$

The scalar product on $H^m(\Omega)$ is defined by

$$(u, v)_m = \sum_{|\alpha| \leq m} (\partial^\alpha u, \partial^\alpha v)_0,$$

with the associated norm

$$||u||_m = (u, u)_m^{1/2} = (\sum_{|\alpha| \leq m} ||\partial^\alpha u||_0^2)^{1/2} ,$$

and the corresponding semi-norm

$$|u|_m = (\sum_{|\alpha| = m} ||\partial^\alpha u||_0^2)^{1/2}.$$

We now define the Sobolev space on the boundary Γ which is necessary to define the integral operators (for details see e.g.Dautray and Lions [15] and Sauter and Schwab [41]). The L^2-norm on Γ is defined similarly to the space $L^2(\Omega)$ and equipped with the norm

$$||u||_0^2 = \int_\Gamma |u(x)|^2 \, ds_x.$$

For simplicity, we assume that there exists a piecewise parameterization of the boundary

$$\chi : \xi \mapsto x, \quad \xi = (\xi_1, \cdots, \xi_{d-1}) \in \mathcal{G} \subset \mathbb{R}^{d-1}, \quad x \in \Gamma\,.$$

The definition of higher order Sobolev spaces on Γ requires the partial derivatives with respect to the parameters ξ

$$\partial^\alpha u(x) = \left(\frac{\partial}{\partial \xi_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial \xi_{d-1}}\right)^{\alpha_{d-1}} u(\chi(\xi_1, \cdots, \xi_{d-1})), \quad x \in \Gamma\,.$$

The Sobolev spaces of order $k \in \mathbb{N}_0$, $k \leq l$ on the boundary is defined as the closure of the space $\{u \in C^\infty(\Gamma) : ||u||_k < \infty\}$ with respect to the norm

$$||u||_k = \left(\sum_{|\alpha| \leq k} ||\partial^\alpha u||_{L^2(\Gamma)}^2\right)^{1/2}.$$

The generalization of the case of the Sobolev spaces of real positive order $s = k + r$, where $k \in \mathbb{N}_0$, $r \in (0, 1)$ is realized by the corresponding Sobolev-Slobodeckii norm

$$||u||_s = \left(||u||_k^2 + |u|_r^2\right)^{1/2}$$

with the half-norm

$$|u|_r = \left(\sum_{|\alpha|=r} \int_\Gamma \int_\Gamma \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x-y|^{d-1+2r}} ds_x \, ds_y\right)^{1/2} .$$

Employing the dual product

$$\langle u, v\rangle = \int_\Gamma u(x)v(x)\, ds_x,$$

we introduce the Sobolev spaces $H^{-s}(\Gamma)$ of negative order for $s \in (0, l]$ as the dual spaces to $H^s(\Gamma)$

$$H^{-s}(\Gamma) = (H^s(\Gamma))',$$

with the norm

$$||u||_{-s} = \sup_{0 \neq v \in H^s(\Gamma)} \frac{\langle u, v\rangle}{||v||_s} .$$

Definition 1.1. *Let $||\varphi||_{k,l} = \sup_{x\in\mathbb{R}^N}(|x|^k + 1)\sum_{|\alpha|\leq l}|D^\alpha\varphi(x)|$ for $k, l \geq 0$ and any multiindex α. Then*

$$\mathcal{S}(\mathbb{R}^N) = \{\varphi \in C^\infty(\mathbb{R}^N)| \quad ||\varphi||_{k,l} < \infty \textit{ for all multi-indices } k, l \in \mathbb{N}_0\}$$

is called the Schwartz space of tempered functions. The dual space of $\mathcal{S}(\mathbb{R}^N)$ denoted by $\mathcal{S}'(\mathbb{R}^N)$ is called the space of tempered distributions.

Let E be a Hilbert space and define

$$LT(\sigma, E) = \{f \in \mathcal{D}'_+(E); e^{-\sigma t} f \in \mathcal{S}'_+(E)\}$$

where $\mathcal{D}'_+(E)$ and $\mathcal{S}'_+(E)$ denote, as usual, the sets of distributions and temperate distributions on $\mathbb{R}$, with values in E and support in $[0, \infty[$. It is clear that $LT(\sigma, E) \subset LT(\sigma', E)$ if $\sigma < \sigma'$.
We denote by $\sigma(f)$ the infimum of all σ such that $f \in LT(\sigma, E)$. We thus have the set of Laplace transformable distributions with values in E given by

$$LT(E) = \underset{\sigma\in\mathbb{R}}{\cup} LT(\sigma, E) \tag{1.1}$$

For $f \in LT(E)$, we define its Fourier-Laplace transform (as in the scalar case) by

$$\hat{f}(\omega) = \mathcal{F}(e^{-\sigma t} f)(\eta)$$

for $\sigma > \sigma(f)$.
For convenience we recall here the main results on the Fourier-Laplace transform. For reference, we state the well-known Paley-Wiener and the Parseval-Plancherel identity. Lemma 1.1 allows to map results of existence and uniqueness obtained in the frequency domain to the time domain(reference), and Lemma 1.2 can be used to deduce mapping properties of the time dependent operators from the mapping properties of the time independent operators.

Lemma 1.1. *(Paley-Wiener) An E-valued function $\hat{f}(\omega)$ is the Fourier-Laplace transform of $f \in LT(E)$ if and only if it is holomorphic in some half planes $C_{\sigma_0} = \{\omega \in \mathbb{C}; Im\omega > \sigma_0\}$ and if it is of temperate growth in some closed half planes of C_{σ_0}. This last condition means that there exist $\sigma_1 > \sigma_0$, $C > 0$ and $k \in \mathbb{N}^*$ such that*

$$||\hat{f}(\omega)||_E \leq C(1 + |\omega|)^k$$

for all ω with $Im\omega \geq \sigma_1$.

Lemma 1.2. *(Parseval theorem) If $f, g \in L^1_{loc}(\mathbb{R}, E) \cap LT(E)$, one has the following formula*

$$\frac{1}{2\pi}\int_{\mathbb{R}+i\sigma}(\hat{f}(\omega), \hat{g}(\omega))_E d\omega = \int_{\mathbb{R}} e^{-2\sigma t}(f(t), g(t))_E dt$$

where $(.,.)_E$ is the hermitian product of E and $\sigma > \max(\sigma(f), \sigma(g))$.

Spatio-temporal Sobolev spaces:

Recall the definition of the $H^s(\mathbb{R}^N)$-norm

$$||u||^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N}(1 + |\xi|^2)^s|\hat{u}(\xi)|^2 d\xi\,.$$

For $\omega \in \mathbb{C} \neq 0$ with $Im(\omega) = \sigma > 0$, the norm

$$||u||^2_{s,\omega,\mathbb{R}^N} = \int_{\mathbb{R}^N}(|\omega|^2 + |\xi|^2)^s|\hat{u}(\xi)|^2 d\xi$$

is equivalent to the $||.||^2_{H^s(\mathbb{R}^N)}$-norm.
For a domain $\Omega \subset \mathbb{R}^3$, the ω-indexed norm in $H^1(\Omega)$ is given by

$$||u||_{1,\omega,\Omega} = (\int_\Omega |\nabla u|^2 + |\omega|^2|u|^2)^{\frac{1}{2}}\,. \tag{1.2}$$

As for the traces of elements of $H^1(\Omega)$, we introduce a partition of unity α_i subordinate to the covering of Γ by open sets B_i. We consider a smooth partition of unity, diffeomorphisms (φ_i) mapping each B_i into the unit cube Q and $B_i \cap \Omega$ into $Q^+ = \{x \in Q, x_3 > 0\}$, thus $B_r \cap \Gamma$ into $\Sigma = \{x \in Q, x_3 = 0\}$.
Now, for f defined on Γ, one sets

$$(\theta_i f)(x') = (\alpha_i f) \circ \varphi_i^{-1}(x', 0) \quad x' \in \Sigma$$

and

$$||f||_{s,\omega,\Gamma} = \left(\sum_{i=1}^{p} \int_{\mathbb{R}^2} (|\omega|^2 + |\xi|^2)^s |\widehat{\theta_i f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} ,$$

where $\widehat{\theta_i f}(\xi)$ denotes the Fourier transform of this function.
According to [5] and the references therein, we can now define the following spaces:

Definition 1.2. *Let $s, \sigma \in \mathbb{R}$ and $s > 0$, $\sigma > 0$ then*

$$\mathcal{H}^s_\sigma(\mathbb{R}^+, X) = \{ f \in \mathcal{L}'(\sigma, X); e^{-\sigma t}\Lambda^s f \in L^2(\mathbb{R}, X) \}$$

where $\widehat{\Lambda^s f}(\omega) = (i\omega)^s \hat{f}(\omega)$. It is equipped with the norm

$$||f||_{\sigma,s,X} = \left(\frac{1}{2\pi} \int_{-\infty+i\sigma}^{+\infty+i\sigma} |\omega|^{2s} ||\hat{f}(\omega)||^2_X d\omega \right)^{\frac{1}{2}} .$$

We also need the following spatio-temporal Sobolev spaces:

Definition 1.3. *[8, p.41] Let $s \in \mathbb{R}$ and $m \in \mathbb{R}$, then*

$$H^s_\sigma(\mathbb{R}^+, H^m(\Gamma)) = \{ u \in LT(H^s(\Gamma)) \quad | \quad ||u||_{s,m,\Gamma} < \infty \} .$$

It is equipped with the norm

$$\|f\|_{s,m,\Gamma} = \left(\frac{1}{2\pi} \int_{-\infty+i\sigma}^{+\infty+i\sigma} |\omega|^{2s} \|\hat{f}(\omega)\|^2_{m,\omega,\Gamma} d\omega \right)^{\frac{1}{2}} .$$

For $m \in \mathbb{N}$ we have

$$H^s_\sigma(\mathbb{R}^+, L^2(\Gamma)) = \mathcal{H}^s_\sigma(\mathbb{R}^+, L^2(\Gamma)) ,$$
$$H^s_\sigma(\mathbb{R}^+, H^m(\Gamma)) = \left\{ f \in H^s_\sigma(\mathbb{R}^+, H^{m-1}(\Gamma)); \nabla f \in H^{s-1}_\sigma(\mathbb{R}^+, H^{m-1}(\Gamma)^3) \right\}$$

and $\|\hat{f}(\omega)\|_{m,\omega,\Gamma}$ is defined recursively by

$$\|\hat{f}(\omega)\|_{0,\omega,\Gamma} = ||\hat{f}(\omega)||_{L^2(\Gamma)}$$
$$\|\hat{f}(\omega)\|^2_{m,\omega,\Gamma} = |\omega|^2 \|\hat{f}(\omega)\|^2_{m-1,\omega,\Gamma} + \|\widehat{\nabla f}(\omega)\|^2_{m-1,\omega,\Gamma} .$$

Now we summarize some of the relevant results of the trace operator.

Lemma 1.3. *(Lemma 1 in[22]) For $\sigma > 0$, there exists a constant $C_\sigma(\Gamma)$ so that $\forall \psi \in H^{\frac{1}{2}}(\Gamma)$ and $\omega \in \{Im(\omega) \geq \sigma\}$, $\exists u \in H^1(\Omega)$ so that*

$$||u||_{1,\omega,\Omega} \leq C_\sigma(\Gamma) ||\psi||_{\frac{1}{2},\omega,\Gamma} . \tag{1.3}$$

Lemma 1.4. *(Lemma 2 in[22]) Let γ_0 denote the trace operator on $H^1(\Omega)$. Then for all $\omega \in \{Im(\omega) \geq \sigma\}$*

$$||\gamma_0 u||_{\frac{1}{2},\omega,\Gamma} \leq C_\sigma(\Gamma) ||u||_{1,\omega,\Omega} .$$

1.2 Discretisation of the Space and Time Domain

Let us assume that Γ is the boundary of a nonempty, open, connected and bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$. The outer normal on $\Gamma := \partial\Omega$ is denoted by n.
If Γ is not polygonal we approximate it by a piecewise polynomial surface and write Γ again for the approximation. For simplicity, we will use here a surface composed of N triangular facets with the following properties

- $\Gamma = \cup_{i=1}^{N} \Gamma_i$
- each element Γ_i is closed with $int(\Gamma_i) \neq \varnothing$
- for distinct $\Gamma_i,\ \Gamma_j \subset \Gamma$ it holds $int(\Gamma_i) \cap int(\Gamma_j) = \varnothing$.

For the time discretization we consider a uniform decomposition of the time interval $[0, \infty)$ into subintervals $I_n = [t_{n-1}, t_n)$ with time step $|I_n| = \Delta t$, such that $t_n = n\Delta t$ $(n = 1, \cdots)$.
We choose a basis $\varphi_1^p, \cdots, \varphi_{N_s}^p$ of the space V_h^p of piecewise polynomial functions of degree p in space (continuous if $p \geq 1$) and a basis $\beta^{1,q}, \cdots, \beta^{N_t,q}$ of the space $V_{\Delta t}^q$ of piecewise polynomial functions of degree of q in time (continuous if $p \geq 1$).
Let $\mathcal{T}_S = T_1, \cdots, T_{N_s}$ be the spatial mesh for Γ and $\mathcal{T}_T = [0, t_1), [t_1, t_2), \cdots, [T_{N_t-1}, T)$ the time mesh for a finite subinterval $[0, T)$.
If the simplest type of meshes $\mathcal{T}_{S,T} = \mathcal{T}_S \times \mathcal{T}_T$ is considered, we can approximate $H_\sigma^s(\mathbb{R}^+, X)$ by the space of piecewise polynomials in space and time. It is simply the tensor product of the approximation spaces in space and time, V_h^p and $V_{\Delta t}^q$, and we write [13, p. 535]

$$V_{h,\Delta t}^{p,q} = V_h^p \otimes V_{\Delta t}^q \, .$$

The approximation space is then

$$\begin{aligned} V_{h,\Delta t}^{p,q} = \{\mathrm{span}(\psi)\},\ &\psi \text{ has compact support on } \mathcal{T}_{S,T} \text{ and } \psi_{|\Gamma_k \times [t_{n-1}, t_n)} \\ &\text{is a polynomial of degree } \leq p \text{ in } x \text{ and} \leq q \text{ in } t\ \forall k \geq 0,\ \forall n \geq 1\} \, . \end{aligned}$$

We write $\gamma^m = \beta^{m,0},\ m = 1, \cdots, N_t$, for piecewise constant functions in time, and $\beta^m = \beta^{m,1},\ m = 1, \cdots, N_t$ for the basis of hat functions for the the space of piecewise linear, continuous functions in time.
The hat functions β^m have support on two time intervals, namely $[t_{m-1}, t_m) \cup [t_m, t_{m+1})$, and are given explicitly by

$$\beta^m(t) = \begin{cases} \frac{1}{\Delta t_m}(t - t_{m-1}) & , \quad t \in [t_{m-1}, t_m) \\ \frac{1}{\Delta t_{m+1}}(t_{m+1} - t) & , \quad t \in [t_m, t_{m+1}) \\ 0 & , \quad t \notin [t_{m-1}, t_m) \cup [t_m, t_{m+1}) \end{cases} \, . \tag{1.4}$$

The indicator functions $\gamma^m = \beta^{m,0}$ have support only in one time interval, namely $[t_{m-1}, t_m)$ and are given explicitly by

$$\gamma^m(t) = \begin{cases} 1 \quad , \quad t \in [t_{m-1}, t_m) \\ 0 \quad , \quad t \notin [t_{m-1}, t_m) \end{cases} . \tag{1.5}$$

We note that we can rewrite all the basis functions in terms of Heaviside functions

$$\begin{aligned}
\gamma^m(t) &= H(t - t_{m-1}) - H(t - t_m) \\
\beta^m(t) &= \frac{1}{\Delta t_m}(t - t_{m-1})\gamma^m(t) + \frac{1}{\Delta t_{m+1}}(t_{m+1} - t)\gamma^{m+1}(t) \\
&= (H(t - t_{m-1}) - H(t - t_m))\frac{t - t_{m-1}}{\Delta t_m} + (H(t - t_m) - H(t - t_{m+1}))\frac{t_{m+1} - t}{\Delta t_{m+1}} \, .
\end{aligned}$$

We further note that we understand the derivatives of γ^m in the distributional sense as the difference of two Dirac distributions

$$\dot{\gamma}^m(t) = \delta(t - t_{m-1}) - \delta(t - t_m) \, .$$

The above expressions will be used throughout this thesis.

2 Retarded Potential Boundary Integral Equation in $\mathbb{R}^3$

Before we consider the sound radiation in the half-space $\mathbb{R}^3_+$ in the next chapters we collect here known results in $\mathbb{R}^3$. The propagation of acoustic waves in a homogeneous medium in $\mathbb{R}^3$ is governed by the scalar wave equation

$$\Box\, u := \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \Delta u = 0\,, \tag{2.1}$$

where u is the acoustic pressure and c is the speed of sound in the considered medium. In the following, we set $c = 1$.

In order to study acoustic scattering problems we consider the following setting:

Let $\Omega := \Omega^e \subset \mathbb{R}^3$ be an unbounded connected domain with bounded complement $\Omega^i = \mathbb{R}^3 \backslash \bar{\Omega}$ and Lipschitz boundary $\Gamma = \partial\Omega^e = \partial\Omega^i$. Suppose that an incident field u^i, propagating in Ω, hits the scatterer Ω^i at a certain time. We assume that the incident field has not reached Ω^i at $t = 0$ and that all functions are causal.

Moreover, after subtracting u^i the following initial conditions hold

$$u(0, x) = \frac{\partial}{\partial t} u(0, x) = 0 \quad \text{ for } x \in \Omega\,. \tag{2.2}$$

The boundary conditions on Γ are given by an operator $\mathcal{B}$ acting on u

$$\mathcal{B}u(t, x) = f(t, x) \quad \text{ in } \mathbb{R}^+ \times \Gamma\,. \tag{2.3}$$

If $\mathcal{B}u = u$ we refer to Ω^i as a soft scatterer and the above problem is called the *Dirichlet problem.* For $\mathcal{B}u = \frac{\partial u}{\partial n} - \frac{\alpha}{c}\frac{\partial u}{\partial t}$ the problem is called the acoustic Robin problem or an absorbing scatterer. α is known as the impedance function of the surface Γ, with $\alpha(x) \geq 0$ for all $x \in \Gamma$. For $\alpha(x) \equiv 0$ we have a hard scatterer and Neumann boundary condition. For the scaterring problem we have

$$f(t, x) = -\mathcal{B}u^{\text{inc}}(t, x)\,.$$

The energy of the total pressure field $u^{\text{tot}} := u + u^{\text{inc}}$ is given by

$$E(t, u^{\text{tot}}) = \frac{1}{2}\int_\Omega |\nabla u^{\text{tot}}(t, x)|^2 + |\dot{u}^{\text{tot}}(t, x)|^2\, dx\,.$$

Note that we do not have to require an explicit radiation condition. The fundamental solution of the scalar wave equation (2.1) is

$$G(s, t, x, y) = \frac{\delta(t - s - |x - y|)}{|x - y|}\,,$$

where δ is the Dirac delta distribution.
According to [45] u admits an integral representation

$$u(t,x) = \frac{1}{4\pi}\int_\Gamma \frac{n_y(x-y)}{|x-y|}\left(\frac{\varphi(\tau,y)}{|x-y|^2} + \frac{\dot{\varphi}(\tau,y)}{|x-y|}\right) ds_y - \frac{1}{4\pi}\int_\Gamma \frac{p(\tau,y)}{|x-y|}\, ds_y \qquad (2.4)$$

for all $(t,x) \in \Omega \times \mathbb{R}^+$ with a *retarded time* argument $\tau := t - |x-y|$, where $\varphi = u^+ - u^- = [u]$ and $p = \frac{\partial u^+}{\partial n} - \frac{\partial u^-}{\partial n} = \left[\frac{\partial u}{\partial n}\right]$ with $v^+ = \lim_{\Omega^e \ni x \to \Gamma} v(x)$, $v^- = \lim_{\Omega^i \ni x \to \Gamma} v(x)$.

Remark 2.1. *If $c \neq 1$ the retarded time argument is $\tau = t - |x-y|/c$.*

Definition 2.1. *Define for $(t,x) \in \mathbb{R}^+ \times (\mathbb{R}^3 \setminus \Gamma)$ the retarded single layer potential by*

$$Sp(t,x) = \frac{1}{4\pi}\int_\Gamma \frac{p(\tau,y)}{|x-y|}\, ds_y$$

and the retarded double layer potential by

$$D\varphi(t,x) = \frac{1}{4\pi}\int_\Gamma \frac{n_y \cdot (x-y)}{|x-y|}\left(\frac{\varphi(\tau,y)}{|x-y|^2} + \frac{\dot{\varphi}(\tau,y)}{|x-y|}\right) ds_y\,.$$

Thus, (2.4) reads

$$u(t,x) = D\varphi(t,x) - Sp(t,x)\,,$$

where $\varphi = u^+ - u^- = [u]$ and $p = \frac{\partial u^+}{\partial n} - \frac{\partial u^-}{\partial n} = \left[\frac{\partial u}{\partial n}\right]$.

Definition 2.2. *We define the time domain or retarded potential boundary integral operators for $x \in \Gamma$ and $t \in \mathbb{R}^+$. The single layer potential is given by*

$$Vp(t,x) = \frac{1}{2\pi}\int_\Gamma \frac{p(\tau,y)}{|x-y|}\, ds_y$$

and its normal derivative with respect to x, the adjoint double layer potential, is

$$\begin{aligned} K'p(t,x) &= \frac{1}{2\pi}\int_\Gamma n_x \cdot \nabla_x \frac{p(\tau,y)}{|x-y|} d\sigma_y \\ &= \frac{1}{2\pi}\int_\Gamma \frac{n_x \cdot (x-y)}{|x-y|}\left(\frac{p(\tau,y)}{|x-y|^2} + \frac{\dot{p}(\tau,y)}{|x-y|}\right) ds_y\,. \end{aligned}$$

The double layer potential is given by

$$\begin{aligned} K\varphi(t,x) &= \frac{1}{2\pi}\int_\Gamma -n_y \cdot \nabla_x \frac{\varphi(\tau,y)}{|x-y|} d\sigma_y \\ &= \frac{1}{2\pi}\int_\Gamma \frac{n_y \cdot (x-y)}{|x-y|}\left(\frac{\varphi(\tau,y)}{|x-y|^2} + \frac{\dot{\varphi}(\tau,y)}{|x-y|}\right) ds_y \end{aligned}$$

and its normal derivative, the so-called hypersingular operator, is

$$W\varphi(t,x) = -\lim_{x' \in \Omega^e \to x} n_x \cdot \nabla_{x'}\left(\frac{1}{2\pi}\int_\Gamma n_y \cdot \nabla_{x'} \frac{\varphi(t-|x'-y|,y)}{|x'-y|}\, ds_y\right)\,.$$

The definitions above differ from the more common one by a factor of two.

Remark 2.2. *Formally it holds e. g. for the single layer potential that*

$$Sp(t,x) = \frac{1}{4\pi} \int_0^\infty \int_\Gamma \frac{\delta(t-s-|x-y|)}{|x-y|} p(s,y)\, d\sigma_y\, ds = \frac{1}{4\pi} \int_\Gamma \frac{p(\tau,y)}{|x-y|}\, d\sigma_y\,.$$

Denote the limits from Ω^e and Ω^i by

$$\begin{aligned}(u)^e &= \lim_{x\to\Gamma,\, x\in\Omega^e} u(x)\,,\\ (u)^i &= \lim_{x\to\Gamma,\, x\in\Omega^i} u(x)\,.\end{aligned}$$

Theorem 2.1 (Jump relations)**.** *Let $x \in \Omega^e$ or $x \in \Omega^i$, then for $\varphi, p \in C^2(\mathbb{R}\times\Gamma)$ there holds*

$$\begin{aligned}2(Sp)^-(t,x) = 2(Sp)^+(t,x) &= Vp(t,x)\\ 2\frac{\partial(Sp)^-}{\partial n}(t,x) &= (I+K')p(t,x)\\ 2\frac{\partial(Sp)^+}{\partial n}(t,x) &= (-I+K')p(t,x)\\ 2(D\varphi)^-(t,x) &= (-I+K)\varphi(t,x)\\ 2(D\varphi)^+(t,x) &= (I+K)\varphi(t,x)\\ 2\frac{\partial(D\varphi)^-}{\partial n}(t,x) = 2\frac{\partial(D\varphi)^+}{\partial n}(t,x) &= W\varphi(t,x)\,.\end{aligned}$$

Proof. See [24] (Lemma 3 and Lemma 4a). □

We introduce the jump across Γ, which is defined as $[u] := u^+ - u^-$, and define the traces $\gamma_0 u = u$ and $\gamma_1 u = \frac{\partial u}{\partial n}$. We can write the above theorem in a more compact way resulting in the well known jump relations

$$\begin{aligned}[\gamma_0 Sp] &= 0 \quad & [\gamma_1 Sp] &= -p\\ [\gamma_0 D\varphi] &= \varphi \quad & [\gamma_1 D\varphi] &= 0\,.\end{aligned}$$

2.1 Retarded Potential Boundary Integral Equations

In this section we focus on integral equations of the first and second kind. See [22] and the references therein for the corresponding analysis for integral equations.
For the *Dirichlet problem*, or *soft scatterer*, due to the Corollary of Theorem 1 (p. 116) in [24], we can represent the solution u of (2.1) using a single layer ansatz for $x \notin \Gamma$

$$u(t,x) = Sp(t,x)$$

with a density function p. The single layer ansatz is continuous across the boundary (Theorem 2.1) so that the indirect approach yields the boundary integral equation

$$Vp(t,x) = 2f(t,x)\,. \tag{2.5}$$

The variational form reads as follows: Find p in a suitable space–time Sobolev space such that for all test functions q

$$\int_0^\infty\int_\Gamma Vp(t,x)\partial_t q(t,x) ds_x\, d_\sigma t = 2\int_0^\infty\int_\Gamma f(t,x)\partial_t q(t,x) ds_x\, d_\sigma t\ , \tag{2.6}$$

where $d_\sigma t = e^{-2\sigma t}dt$. We will introduce suitable Sobolev spaces in Chapter 3.
On the other hand, we can use the representation formula (2.4) for given boundary data $u = f$ on Γ and obtain with Theorem 2.1 and $p := \gamma_1 u$ the direct formulation

$$Vp = (K - I)f\,. \tag{2.7}$$

The variational form is to find p such that for all test functions q the following holds:

$$\int_0^\infty\int_\Gamma Vp(t,x)\partial_t q(t,x) ds_x\, d_\sigma t = \int_0^\infty\int_\Gamma (K-I)f(t,x)\partial_t q(t,x) ds_x\, d_\sigma t \tag{2.8}$$

For the *Neumann problem* or *hard scatterer*, we can represent u using the double layer potential by some density function φ, i.e. $u = D\varphi$. The direct approach with a given normal derivative $\partial_n u = f$ on the boundary Γ yields

$$W\varphi = (I + K')f\,, \tag{2.9}$$

where $\varphi = \gamma_0 u$.
The single layer potential ansatz leads to the indirect formulation

$$(I - K')\varphi = -2f\,. \tag{2.10}$$

The variational formulations of the Neumann problem are to find φ resp. p, such that

$$\int_0^\infty\int_\Gamma W\varphi(t,x)\partial_t\psi(t,x) ds_x\, d_\sigma t = \int_0^\infty\int_\Gamma (I+K')f(t,x)\partial_t\psi(t,x) ds_x\, d_\sigma t\,, \tag{2.11}$$

$$\int_0^\infty\int_\Gamma (I-K')p(t,x)\partial_t q(t,x) ds_x\, d_\sigma t = -2\int_0^\infty\int_\Gamma f(t,x)\partial_t q(t,x) ds_x\, d_\sigma t \tag{2.12}$$

for all test functions.

2.2 The Marching On In Time Method

Let us now summarize the fully discrete schemes as presented in the thesis of E. Ostermann [40, Section 2.3.2] [35] [44]. The single layer potential ansatz using piecewise constant test and trial functions results in the following algebraic system with $n = 1, \ldots$

$$\sum_{m=1}^{n} V^{n-m} p^m = 2(f^{n-1} - f^n)\,,$$

which yields

$$V^0 p^n = 2(f^{n-1} - f^n) - \sum_{m=1}^{n-1} V^{n-m} p^m\,.$$

For (2.8) we obtain

$$V^0 p^n = I(f^n - f^{n-1}) + \sum_{m=1}^{n} K^{n-m} f^m - \sum_{m=1}^{n-1} V^{n-m} p^m$$

where V^l is given in [40, (2.25)], K^l in [40, (2.28)] with $l := n-m$ are given explicitly in [40, Section 2.3.2] and I denotes the corresponding mass matrix. The direct approach for the Neumann problem (2.9) for piecewise linear trial and piecewise constant test functions in time yields

$$W^0 \phi^n = \frac{\Delta t}{2} I(f^{n-1} + f^n) + \sum_{m=1}^{n} (K^{n-m})^T f^m - \sum_{m=1}^{n-1} W^{n-m} \phi^m$$

where W^l is defined as in [40, (2.27)] and K^l is given in [40, (2.27)] .
Similary, for (2.12) with piecewise constant trial and test functions in time we get

$$(-\Delta t\; I + (K')^0)\varphi^n = 2F^n - \sum_{m=1}^{n-1} (K')^{n-m} \varphi^m\;.$$

The above fully discrete systems involve the computation of a series of matrices, that are sparsely populated because the light cone integration domain E_l restricts the number of interacting elements per time step.
Note that as observed in [40] the computation of each matrix only depends on the time difference. Furthermore (see [40, Section 2.3.2]) for bounded surfaces Γ the matrices V^{n-m} resp. W^{n-m}, K'^{n-m} vanish whenever the time difference $l := n - m$ satisfies

$$l > \left[\frac{\mathrm{diam}\Gamma}{\Delta t}\right]\,.$$

Now, as shown in [40] for each of the above equations we obtain an MOT scheme of the form

$$A^0 x^n = f^n - \sum_{m=\max(1,n-\hat{n})}^{n-1} A^{n-m} x^m =: b^n\,. \tag{2.13}$$

with

$$b^n = f^n - \sum_{m=1}^{n-1} A^{n-m} x^m, \qquad n \leq \hat{n}\,,$$

and

$$b^n = f^n - \sum_{m=n-\hat{n}}^{n-1} A^{n-m} x^m = f^n - \sum_{m=1}^{\hat{n}} A^{\hat{n}-m+1} x^{m+(n-\hat{n})-1} \quad \text{for} \quad n > \hat{n}.$$

The abstract MOT scheme can be summarized as follows ([40, Algorithm 2.1]):

for $n = 1, \dots$ **do**
- **if** $n > \left[\frac{\mathrm{diam}\Gamma}{\Delta t}\right]$ **then**
  - Domain of influence has passed the body;
  - No more matrix computation needed;
- **else**
  - Allocate storage for basic Galerkin matrix G^{n-1} ;
  - Compute G^{n-1};
  - Compose the new retarded matrices;
  - Delete basic Galerkin entries that are not needed in the next time step;
- **end**
- Compute right hand side by matrix vector multiplication;
- Solve the system of linear equations ;
- Store new solution vector

end

Algorithm 1: Time Stepping Algorithm

The most expensive part of the MOT scheme is the computation of the matrix entries, although the resulting matrices are sparse.

The computation of an entry in the Galerkin matrix, which is an integral of the type

$$G_{ij}^{l,\nu} := \iint\limits_{E_l} k_\nu(x-y)\varphi_i(y)\varphi_j(x)\, ds_y\, ds_x\,, \tag{2.14}$$

where $k_\nu(x-y) = |x-y|^\nu$ and $\nu \geq -1$, is in detail described in [40, Chapter 4]. The discrete light cone integration domain is given by

$$E_l := \{(x,y) \in \Gamma \times \Gamma \text{ s.t. } t_l \leq |x-y| \leq t_{l+1}\}\ .$$

The basic idea is to rewrite (2.14) on triangles T_i, T_j of the triangulation Γ_h of Γ as

$$\begin{aligned} G_{ij}^{l,\nu} &= \int\limits_{\Gamma_h} \int\limits_{\Gamma_h \cap E(x)} k_\nu(x-y)\varphi_i(y)\varphi_j(x)\, ds_y\, ds_x \\ &= \sum_{T_i, T_j \in \Gamma_h} \int\limits_{T_i} \int\limits_{T_j \cap E(x)} k_\nu(x-y)\varphi_i(y)\varphi_j(x)\, ds_y\, ds_x\,, \end{aligned}$$

where $E(x) := B_{t_{l+1}}(x) \setminus B_{t_l}(x)$ is the so called domain of influence of the point x. $B_r(x)$ denotes the ball of radius r and center x. In [40] the hp-quadrature to compute $G_{ij}^{l,\nu}$ is introduced and analysed.
In Chapter 5 below we present our benchmark computations for the half-space case, discussing e.g. sound radiation above the sound hard road. the results are obtained by applying MOT together with a space-time discretization of the second kind boundary integral equation with the normal derivative of the double layer potential on the surface of the scatterer (tyre). The computations with our extended code use Ostermann's hp-quadrature and are written based on the software package MaiProgs [33].

3 Retarded Potential Boundary Integral Equations in the Half-Space

3.1 The Transient Half-Space Green's Function for an Absorbing Plane

In this chapter we report from Ochmann [38] a model problem for the interaction of a linear sound wave with a partially absorbing surface. Corresponding Green's functions in both frequency and time–domain have been derived in [38] [39].
For small amplitudes the relationship between sound pressure u and the normal velocity v_n of the surface is linear. Their ratio is the acoustic impedance and it is given by:

$$Z = (\frac{u}{v_n})_{|\Gamma} .$$

For simplicity we consider $Z \in \mathbb{R}$ constant and define

$$\alpha_\infty = \frac{\rho_0 c}{Z} .$$

We take $c = 1$. Here ρ_0 is the density of the absorbing medium. α_∞ is the specific acoustic admittance of the surface. A rigid plane has the admittance $\alpha_\infty = 0$.
We consider an acoustic boundary condition of Robin type:

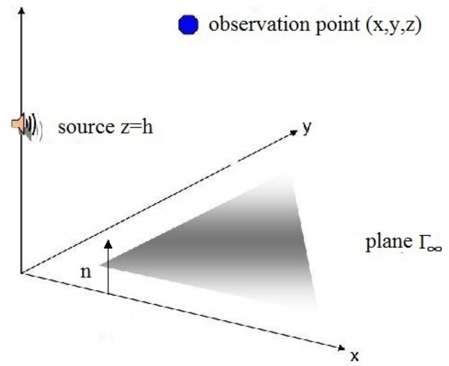
$$\frac{\partial u}{\partial n} - \alpha_\infty \frac{\partial u}{\partial t} = 0 ,$$

where n is the unit normal vector to the boundary which points into Ω^e.
We would like to determine a Green's function for the wave equation with this boundary condition. The Green's function represents the acoustic response in half-space to a Dirac mass. It corresponds to a function G, which depends on the admittance α_∞, on a fixed source point $P \in \mathbb{R}^3_+$, and on an observation point $Q \in \mathbb{R}^3_+$, where $\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$ is the half-space.
The Green's function is defined in the sense of distributions in the half-space $\mathbb{R}^3_+$ by placing a Dirac impulse $\delta(t, x_1, x_2, x_3 - h)$ at time $t = 0$ and at the location $P = (0, 0, h)$ on the right-hand side of the wave equation.
It is therefore a solution to the following problem:

$$\frac{\partial^2 G}{\partial t^2} - \Delta G = \delta(t, x_1, x_2, x_3 - h) \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^3_+$$

with the boundary condition on an infinite plane $\Gamma_\infty = \{(x_1, x_2, x_3) \in \mathbb{R}^3 / x_3 = 0\}$:

$$\frac{\partial G}{\partial x_3} - \alpha_\infty \frac{\partial G}{\partial t} = 0\,. \tag{3.1}$$



In the frequency domain this can be written as the Helmholtz equation

$$\Delta \hat{G} + \omega^2 \hat{G} = \delta(x_1, x_2, x_3 - h)$$

with the following condition on Γ_∞

$$\frac{\partial \hat{G}}{\partial x_3} + \beta_\infty \hat{G} = 0\,. \tag{3.2}$$

Here $\beta_\infty = i\omega\alpha_\infty$.
In addition a radiation condition is imposed at infinity.
Using an ansatz going back to Sommerfeld, Ochmann tries to find a solution in the form

$$\hat{G} = \hat{g}(h) + \hat{g}(-h) + \int_{-\infty}^{-h} a(\eta)\hat{g}(\eta)\, d\eta \tag{3.3}$$

with the unknown function $a(\eta)$ and the free-space Green's function

$$\hat{g}(h) = \frac{1}{4\pi} \frac{e^{i\omega\sqrt{x_1^2+x_2^2+(x_3-h)^2}}}{\sqrt{x_1^2+x_2^2+(x_3-h)^2}}. \tag{3.4}$$

The second and third terms on the right-hand side of $\hat{G}$ represent the field reflected by the plane Γ_∞.
To determine $a(\eta)$ one inserts the ansatz for $\hat{G}$ in the equation (3.2). First we calculate the normal derivative of $\hat{G}$:

$$\frac{\partial \hat{G}}{\partial x_3}_{|x_3=0} = -\int_{-\infty}^{-h} a(\eta) \frac{\partial}{\partial \eta}(\hat{g}(\eta))\, d\eta$$

using

$$\frac{\partial \hat{g}}{\partial x_3} = -\frac{\partial \hat{g}}{\partial \eta}.$$

By performing an integration by parts, one obtains

$$\begin{aligned} I &= \int_{-\infty}^{-h} a(\eta) \frac{\partial}{\partial \eta} (\frac{e^{i\omega r_\eta}}{r_\eta})\, d\eta \\ &= \frac{a(-h)}{r_h} e^{i\omega r_h} - \int_{-\infty}^{-h} \frac{\partial a(\eta)}{\partial \eta} (\frac{e^{i\omega r_\eta}}{r_\eta})\, d\eta\,, \end{aligned}$$

assuming that $\frac{a(\eta)}{r_\eta} e^{ikr_\eta} \xrightarrow[\eta \to -\infty]{} 0$.

Here, setting $r_\eta = \sqrt{x_1^2 + x_2^2 + \eta^2}$ and $r_h = \sqrt{x_1^2 + x_2^2 + h^2}$ and substituting $\hat{G}$ and $\frac{\partial \hat{G}}{\partial x_3}$ into the boundary condition (3.2), one obtains

$$\frac{a(-h)}{r_h} e^{i\omega r_h} - \int_{-\infty}^{-h} \frac{\partial a(\eta)}{\partial \eta} (\frac{e^{i\omega r_\eta}}{r_\eta})\, d\eta - \frac{2\beta_\infty}{r_h} e^{i\omega r_h} - \beta_\infty \int_{-\infty}^{-h} a(\eta) \frac{e^{i\omega r_\eta}}{r_\eta}\, d\eta = 0\,.$$

This equation holds provided that

$$\begin{cases} a(-h) - 2\beta_\infty = 0 \\ \frac{\partial a(\eta)}{\partial \eta} + \beta_\infty a(\eta) = 0\,. \end{cases}$$

The solution of these equations is

$$a(\eta) = 2\beta_\infty e^{-\beta_\infty(\eta + h)}\,. \tag{3.5}$$

By substituting equation (3.5) into (3.3) we obtain

$$\hat{G} = \hat{g}(h) + \hat{g}(-h) + 2\beta_\infty e^{-\beta_\infty h} \int_{-\infty}^{-h} e^{-\beta_\infty \eta} \hat{g}(\eta)\, d\eta\,. \tag{3.6}$$

Remark 3.1. *We have assumed that:* $\frac{a(\eta)}{r_\eta} e^{i\omega r_\eta} \xrightarrow[\eta \to -\infty]{} 0$. *From equation* (3.5)*, we see that this is satisfied only if* $\mathrm{Re}(\beta_\infty) \leq 0$. *It holds in particular if* $\alpha_\infty \in \mathbb{R}$ *and* $\mathrm{Im}(\omega) \geq 0$.

Starting from the expression for the half-space Green's function in the frequency domain

$$\hat{G} = \frac{e^{i\omega r(h)}}{4\pi r(h)} + \frac{e^{i\omega r(-h)}}{4\pi r(-h)} + 2\beta_\infty e^{-\beta_\infty h} \int_{\infty}^{-h} e^{-\eta \beta_\infty} \frac{e^{i\omega r(\eta)}}{4\pi r(\eta)} d\eta$$

with $r(h) = \sqrt{x_1^2 + x_2^2 + (x_3 - h)^2}$, G is obtained from $\hat{G}$ by the inverse Fourier transform :

$$G = F^{-1}\hat{G} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}(\omega) e^{-i\omega t} d\omega.$$

In summary, the Green's function over an infinite plane with the boundary condition (3.1) can be represented as

$$G = \frac{\delta(t - r(h))}{4\pi r(h)} + \frac{\delta(t - r(-h))}{4\pi r(-h)} + F^{-1}\hat{\Sigma}\,,$$

where

$$\hat{\Sigma} = 2\beta_\infty e^{-\beta_\infty h} \int_\infty^{-h} e^{-\eta\beta_\infty} \frac{e^{i\omega r(\eta)}}{4\pi r(\eta)} d\eta \,.$$

Here one uses the known transform

$$F(\frac{\delta(t - r(h))}{4\pi r(h)}) = \frac{e^{i\omega r(h)}}{4\pi r(h)} \,.$$

It remains to calculate the inverse Fourier transform of the term $\hat{\Sigma}$:

$$\Sigma = F^{-1}(\hat{\Sigma}) = \frac{\alpha_\infty}{2\pi} \int_{-\infty}^{-h} \frac{1}{r(\eta)} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} i\omega e^{i\omega(r(\eta) - \alpha_\infty(\eta + h))} e^{-i\omega t} \, d\omega \right) d\eta \,.$$

Setting

$$I(\eta, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} i\omega e^{i\omega(r(\eta) - \alpha_\infty(\eta + h))} e^{-i\omega t} \, d\omega$$

one considers the Fourier integral

$$J(\eta, t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(r(\eta) - \alpha_\infty(\eta + h))} e^{-i\omega t} \, d\omega$$

with

$$I(\eta, t) = \frac{\partial J(\eta, t)}{\partial t}.$$

Because of

$$J(\eta, t) = -\delta\left(t - r(\eta) + \alpha_\infty(h + \eta)\right) \tag{3.7}$$

and by introducing the quantity

$$\frac{\partial \Sigma_\phi}{\partial t} = \Sigma$$

one gets after interchanging integration and differentiation

$$\begin{aligned} \Sigma_\phi &= \frac{\alpha_\infty}{2\pi} \int_{-\infty}^{-h} \frac{1}{r(\eta)} J(\eta, t) d\eta \\ &= \frac{-\alpha_\infty}{2\pi} \int_{-\infty}^{-h} \frac{1}{r(\eta)} \delta\left(t - r(\eta) + \alpha_\infty(h + \eta)\right) d\eta. \end{aligned}$$

Hence substituting

$$\tau = r(\eta) - \alpha_\infty(h + \eta)$$

with

$$\begin{aligned} \frac{d\tau}{d\eta} &= \frac{d}{d\eta}\left(r(\eta) - \alpha_\infty(h + \eta)\right) \\ &= \left(\frac{d}{d\eta} r(\eta) - \alpha_\infty\right) \\ &= -\frac{(x_3 - \eta)}{r(\eta)} - \alpha_\infty \end{aligned}$$

and

$$\frac{d\eta}{d\tau} = \frac{-r(\eta)}{(x_3 - \eta) + \alpha_\infty r(\eta)} \,,$$

one has

$$\begin{aligned}
\Sigma_\phi &= \frac{\alpha_\infty}{2\pi}\int_{r(-h)}^{\infty} \frac{\delta(t-\tau)}{r(\eta)}\frac{d\eta}{d\tau}d\tau \\
&= \frac{-\alpha_\infty}{2\pi}\int_{r(-h)}^{\infty} \frac{\delta(t-\tau)}{(x_3-\eta)+\alpha_\infty r(\eta)}d\tau \\
&= \frac{-\alpha_\infty}{2\pi}\int_{-\infty}^{\infty} \frac{\delta(t-\tau)H(\tau-r(-h))}{(x_3-\eta)+\alpha_\infty r(\eta)}d\tau.
\end{aligned}$$

The next step is to write the denominator in terms of τ only.

$$\begin{aligned}
((x_3-\eta)+\alpha_\infty r(\eta))^2 &= \alpha_\infty^2 r(\eta)^2 + 2\alpha_\infty r(\eta)(x_3-h) + (x_3-\eta)^2 \\
&= \alpha_\infty^2 r(\eta)^2 + 2\alpha_\infty r(\eta)(x_3-h) + \alpha_\infty^2(x_3-\eta)^2 - (\alpha_\infty^2-1)(x_3-\eta)^2 \\
&= \alpha_\infty^2 r(\eta)^2 + 2\alpha_\infty r(\eta)(x_3-h) + \alpha_\infty^2(x_3-\eta)^2 + (\alpha_\infty^2-1)(R^2-r(\eta)^2) \\
&= r(\eta)^2 + 2\alpha_\infty r(\eta)(x_3-h) + \alpha_\infty^2(x_3-\eta)^2 + (\alpha_\infty^2-1)R^2 \\
&= (r(\eta)+\alpha_\infty(x_3-\eta))^2 + (\alpha_\infty^2-1)R^2 \\
&= (\tau+\alpha_\infty(x_3+h))^2 + (\alpha_\infty^2-1)R^2\,.
\end{aligned}$$

Hence,

$$(x_3-\eta)+\alpha_\infty r(\eta) = \sqrt{(\tau+\alpha_\infty(x_3+h))^2+(\alpha_\infty^2-1)R^2}\,.$$

Inserting this expression into the definition of Σ_ϕ one obtains

$$\begin{aligned}
\Sigma_\phi &= \frac{-\alpha_\infty}{2\pi}\int_{-\infty}^{\infty} \frac{\delta(t-\tau)H(\tau-r(-h))}{\sqrt{(\tau+\alpha_\infty(z+h))^2+(\alpha_\infty^2-1)R^2}}d\tau \\
&= \frac{-\alpha_\infty}{2\pi}\frac{H(t-r(-h))}{\sqrt{(t+\alpha_\infty(x_3+h))^2+(\alpha_\infty^2-1)R^2}}\,.
\end{aligned}$$

This gives a closed analytical representation of the Green's function for an absorbing half-space

$$G = \frac{\delta(t-r(h))}{4\pi r(h)} + \frac{\delta(t-r(-h))}{4\pi r(-h)} + \Sigma \tag{3.8}$$

with

$$\Sigma = \frac{-\alpha_\infty}{2\pi}\frac{\partial}{\partial t}\frac{H(t-r(-h))}{\sqrt{(t+\alpha_\infty(x_3+h))^2+(\alpha_\infty^2-1)R^2}}$$

and $\quad R^2 = \sqrt{x_1^2+x_2^2}\,.$

3.2 Integral Representation and Time-Domain Boundary Integral Equations for an Absorbing Half-Space

We consider the direct scattering problem of linear acoustic waves in an exterior Lipschitz domain $\Omega^e = \mathbb{R}^3_+\backslash\Omega^i$ (see Figure 3.1), Ω^i is bounded and the incident field u_I is

known. The goal is to find the scattered field u as a solution to the wave equation in the exterior open and connected domain Ω^e, such that the total field u_T, decomposed as $u_T = u_I + u$, satisfies a homogeneous impedance boundary condition on the boundary $\Gamma \cup \Gamma_\infty$.
Here the boundary of Ω^i in $\mathbb{R}^3_+$ is denoted by Γ, while Γ_∞ denotes $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}$. Now, the unit normal n points out of Ω^i and on Γ_∞ into Ω^e, (see figure).
Thus the initial and boundary problem for the wave equation is:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \Delta u &= 0 \quad \text{in } \mathbb{R}^+ \times \Omega^e \\ u(0,x) &= \frac{\partial u}{\partial t}(0,x) = 0 \\ \frac{\partial u}{\partial n} - \alpha \frac{\partial u}{\partial t} &= f \quad \text{in } \mathbb{R}^+ \times \Gamma \\ \frac{\partial u}{\partial n} - \alpha_\infty \frac{\partial u}{\partial t} &= 0 \quad \text{on } \mathbb{R}^+ \times \Gamma_\infty\,. \end{aligned} \tag{3.9}$$

The geometry for the boundary value problem (3.9) is depicted in Figure 3.1.

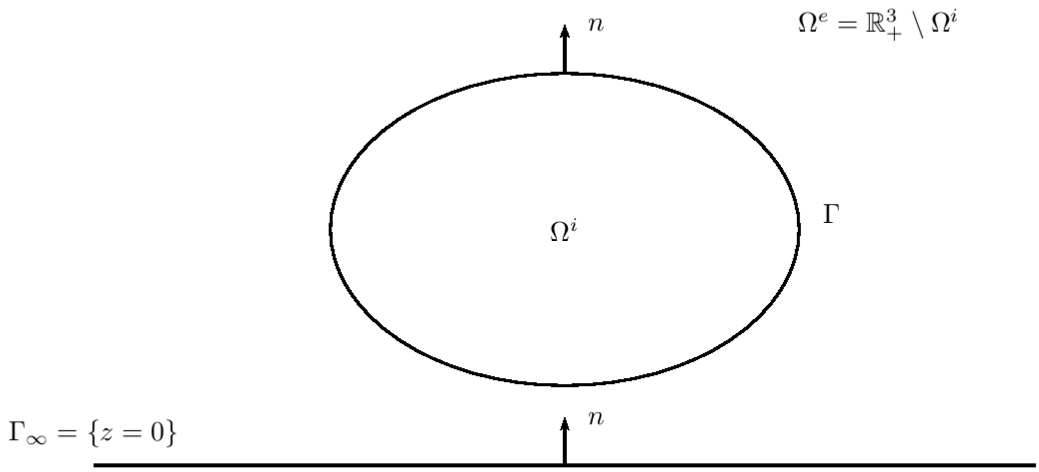


Figure 3.1: Geometry of the boundary value problem (3.9)

3.2.1 Integral Representation

We are interested in expressing the solution u of the direct scattering problem (3.9) by means of an integral representation formula over Γ (See Becache [8] for exterior domains in $\mathbb{R}^3$).

Theorem 3.1. *Let $u \in L^2(\mathbb{R}^+, H^1(\Omega)) \cap H_0^1(\mathbb{R}^+, L^2(\Omega))$ be the solution of (3.9) for a Lipschitz boundary Γ. Then it holds in the sense of distributions:*

$$\begin{cases} u(t,x) = \displaystyle\int_{\mathbb{R}^+\times\Gamma} \frac{\partial G}{\partial n_y}(t-\tau,x,y)u(\tau,y)d\tau ds_y \\ \qquad - \displaystyle\int_{\mathbb{R}^+\times\Gamma} G(t-\tau,x,y)\frac{\partial u}{\partial n}(\tau,y)d\tau ds_y \end{cases} \qquad x \in \Omega^e, t \in \mathbb{R}^+\,,$$

with G given in (3.8).

Remark 3.2. *If $x \in \Omega^i$ we set $u(t,x) = 0$.*

Proof. We define an extension of u to $\mathbb{R}^+ \times \mathbb{R}^3$:

$$\tilde{u} = \begin{cases} u(t,x) & \text{in} \quad \mathbb{R}^+ \times \Omega^e \\ 0 & \text{otherwise.} \end{cases}$$

The proof is divided into 3 steps.
1. Step: We show that $\tilde{u}$ satisfies the following inhomogeneous wave equation:

$$-\Box\tilde{u} = \frac{\partial u}{\partial n}(t,x)\delta_\Gamma + u(t,x)\delta'_\Gamma + \frac{\partial u}{\partial n}(t,x)\delta_{\Gamma_\infty} + u(t,x)\delta'_{\Gamma_\infty}\,, \tag{3.10}$$

where δ_Γ is the delta distribution on Γ, and the distribution δ'_Γ is defined as follows:

$$\langle u\delta'_\Gamma, \theta\rangle = -\int_{\mathbb{R}^+\times\Gamma} u\frac{\partial\theta}{\partial n}\; dt\, ds_x,$$

and

$$\langle u\delta'_{\Gamma_\infty}, \theta\rangle = -\int_{\mathbb{R}^+\times\Gamma_\infty} u\frac{\partial\theta}{\partial n}\; dt\, ds_x\,.$$

Let $\theta \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^3_+)$. Then we have:

$$\langle\Box\tilde{u},\theta\rangle = \langle\tilde{u},\Box\theta\rangle = \int_{\mathbb{R}^+\times\Omega^e} u(t,x)\left(\frac{\partial^2\theta}{\partial t^2} - \triangle\theta\right)(t,x)dt\,dx\,.$$

Since $\tilde{u} = 0$ in Ω^i and $\tilde{u} = 0$ locally integrable it is enough to integrate over Ω^e, where $\tilde{u} = u$.
We now integrate by parts in time

$$\int_{\mathbb{R}^+\times\Omega^e} u(t,x)\frac{\partial^2\theta}{\partial t^2}\; dtdx = \int_{\mathbb{R}^+\times\Omega^e} \frac{\partial^2 u}{\partial t^2}\theta\; dtdx + \int_{\Omega^e}\left[\frac{\partial\theta}{\partial t}u - \theta\frac{\partial u}{\partial t}\right]_0^\infty dx\,.$$

In space we apply Green's second theorem:

$$\begin{aligned} \int_{\mathbb{R}^+\times\Omega^e} -u(t,x)\triangle\theta\; dtdx &= -\int_{\mathbb{R}^+\times\Omega^e} \triangle u\theta\; dtdx + \int_{\mathbb{R}^+\times\partial\Omega^e} (u\partial_n\theta - \theta\partial_n u)\, dtds_x \\ &= -\int_{\mathbb{R}^+\times\Omega^e} \triangle u\theta\; dtdx + \int_{\mathbb{R}^+\times\Gamma} (u\partial_n\theta - \theta\partial_n u)\, dtds_x \\ &\qquad + \int_{\mathbb{R}^+\times\Gamma_\infty} (u\partial_n\theta - \theta\partial_n u)\, dtds_x\,. \end{aligned}$$

This implies

$$
\begin{aligned}
\langle \Box \tilde{u}, \theta \rangle = \langle \tilde{u}, \Box \theta \rangle &= \int_{\mathbb{R}^+ \times \Omega^e} u(t,x) \left(\frac{\partial^2 \theta}{\partial t^2} - \triangle \theta \right) (t,x) dt dx \\
&= \int_{\mathbb{R}^+ \times \Omega^e} u(t,x) \frac{\partial^2 \theta}{\partial t^2} dt dx - \int_{\mathbb{R}^+ \times \Omega^e} u(t,x) \triangle \theta dt dx \\
&= \int_{\mathbb{R}^+ \times \Omega^e} \frac{\partial^2 u}{\partial t^2} \theta \; dt dx + \int_{\Omega^e} \left[\frac{\partial \theta}{\partial t} u - \theta \frac{\partial u}{\partial t} \right]_0^\infty dx \\
&- \int_{\mathbb{R}^+ \times \Omega^e} \triangle u \theta \; dt dx + \int_{\mathbb{R}^+ \times \Gamma} (u \partial_n \theta - \theta \partial_n u) \, dt ds_x \\
&+ \int_{\mathbb{R}^+ \times \Gamma_\infty} (u \partial_n \theta - \theta \partial_n u) \, dt ds_x \\
&= \int_{\mathbb{R}^+ \times \Omega^e} \left(\frac{\partial^2 u}{\partial t^2} - \triangle u \right) \theta dx dt + \int_{\Omega^e} \left[\frac{\partial \theta}{\partial t} u - \theta \frac{\partial u}{\partial t} \right]_0^\infty dx \\
&+ \int_{\mathbb{R}^+ \times \Gamma} (u \partial_n \theta - \theta \partial_n u) \, ds_x dt + \int_{\mathbb{R}^+ \times \Gamma_\infty} (u \partial_n \theta - \theta \partial_n u) \, ds_x dt \, .
\end{aligned}
$$

By the assumptions on u and θ the second integral vanishes. It remains:

$$
\begin{aligned}
\langle \Box \tilde{u}, \theta \rangle = &\int_{\mathbb{R}^+ \times \Omega^e} \left(\frac{\partial^2 u}{\partial t^2} - \triangle u \right) \theta \; dx dt + \int_{\mathbb{R}^+ \times \Gamma} (u \partial_n \theta - \theta \partial_n u) \, ds_x dt \\
&+ \int_{\mathbb{R}^+ \times \Gamma_\infty} (u \partial_n \theta - \theta \partial_n u) \, ds_x dt \, .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
-\langle \Box \tilde{u}, \theta \rangle = &\int_{\mathbb{R}^+ \times \Omega^e} (-\Box u) \, \theta \; dx dt + \int_{\mathbb{R}^+ \times \Gamma} \left(-u \frac{\partial \theta}{\partial n} + \frac{\partial u}{\partial n} \theta \right) \; ds_x dt \\
&- \int_{\mathbb{R}^+ \times \Gamma_\infty} (u \partial_n \theta - \theta \partial_n u) \, ds_x dt \, .
\end{aligned}
$$

We conclude

$$
-\Box \tilde{u} = -\Box u \; 1_{\Omega^e} + \frac{\partial u}{\partial n} \delta_\Gamma + u \delta_\Gamma' + \frac{\partial u}{\partial n} \delta_{\Gamma_\infty} + u \delta_{\Gamma_\infty}' \, ,
$$

where 1_{Ω^e} is the characteristic function of Ω^e. By assumption, u is a solution of $\Box w = 0$. So:

$$
-\Box \tilde{u} = -\underbrace{\Box u}_{=0} \; 1_{\Omega^e} + \frac{\partial u}{\partial n} \delta_\Gamma + u \delta_\Gamma' + \frac{\partial u}{\partial n} \delta_{\Gamma_\infty} + u \delta_{\Gamma_\infty}'
$$

which is (3.10). This proves (3.10).

<u>2. Step</u>: By (3.10), we observe that for $\phi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^3)$

$$
\begin{aligned}
\langle 1_{\mathbb{R}^+ \times \Omega^e} u, \Box \phi \rangle - \langle 1_{\mathbb{R}^+ \times \Omega^e} \Box u, \phi \rangle &= \langle \Box (1_{\mathbb{R}^+ \times \Omega^e} u), \phi \rangle - \langle 1_{\mathbb{R}^+ \times \Omega^e} \Box u, \phi \rangle \\
&= -\langle \frac{\partial v_e}{\partial n} \delta_\Gamma + v_e \delta_\Gamma' + \frac{\partial v_e}{\partial n} \delta_{\Gamma_\infty} + v_e \delta_{\Gamma_\infty}', \phi \rangle \, . \qquad (3.11)
\end{aligned}
$$

Since $\Box u = 0$, the left hand side is equal to $\langle 1_{\mathbb{R}^+ \times \Omega^e} u, \Box \phi \rangle$. While this identity extends to $\phi \in C^\infty(\mathbb{R}^+ \times \bar{\Omega}^e)$, provided ϕ is compactly supported in $[0, \infty)$, we would like to substitute $\phi = G$, the Green's function for the absorbing half-space (3.8). In this case we have for all $(t,x) \in \mathbb{R}^+ \times \Omega^e$:

$$\begin{aligned}\langle 1_{\mathbb{R}^+\times\Omega^e} u, \Box G\rangle &= \langle 1_{\mathbb{R}^+\times\Omega^e} u, \delta_t\delta_x\rangle \\ &= (1_{\mathbb{R}^+\times\Omega^e} u)(t,x) = u(t,x)\,,\end{aligned}$$

and (3.11) is the representation formula.
It remains to show that (3.11) holds for $\phi(y,\tau) = G(x,y,t,\tau)$, given $(t,x) \in \mathbb{R}^+ \times \Omega^e$. To do so, we fix $\tilde{\chi} \in C^\infty$ so that $\tilde{\chi} = 1$ near the singularities of G: I.e. for given $\delta > 0$, we set

$$I = \{||x-y| - (t-\tau)| < \delta\} \cup \{||x-y'| - (t-\tau)| < \delta\}\,.$$

We set

$$\begin{cases}\tilde{\chi}(x,y,t,\tau) = 1 & \text{when} \quad (x,y,t,\tau) \in I \\ \tilde{\chi}(x,y,t,\tau) = 0 & \text{when} \quad ||x-y| - (t-\tau)| > 2\delta \text{ and } ||x-y'| - (t-\tau)| > 2\delta\,,\end{cases}$$

and a suitable smooth function in between.
It is easy to check that (3.11) extends to $\phi = (1-\tilde{\chi})G \in C^\infty(\mathbb{R}^+ \times \Omega^e)$, i.e

$$\int_{\mathbb{R}^+\times\Omega^e} u\Box((1-\tilde{\chi})G) = -\langle \frac{\partial u}{\partial n}\delta_\Gamma + u\delta'_\Gamma + \frac{\partial u}{\partial n}\delta_{\Gamma_\infty} + u\delta'_{\Gamma_\infty}, (1-\tilde{\chi})G\rangle\,.$$

It only remains to consider $\phi = \tilde{\chi}G$. Assume that δ is small enough so that $dist(x, \Gamma \cup \Gamma_\infty) > 2\delta$. We let $\chi_2 \in C^\infty(\mathbb{R}^+ \times \Omega^e)$ defined as

$$\begin{cases}\chi_2 = 1 & \text{when} \quad 0 < dist(y, \Gamma\cup\Gamma_\infty) < \delta \\ \chi_2 = 0 & \text{when} \quad dist(x, \Gamma\cup\Gamma_\infty) > 2\delta\,,\end{cases}$$

and a suitable smooth function in between.
We write

$$\phi = \tilde{\chi}G = \chi_2\tilde{\chi}G + (1-\chi_2)\tilde{\chi}G\,.$$

The second term vanishes near $\Gamma \cup \Gamma_\infty$. We note

$$\begin{aligned}\langle (1-\chi_2)\tilde{\chi}G, \Box u\rangle &= \langle \Box\left((1-\chi_2)\tilde{\chi}G\right), u\rangle \\ &= \int_{I\cap\{dist(y,\Gamma\cup\Gamma_\infty)>2\delta\}} u\Box G\, ds_y d\tau \\ &\qquad + \int_{I^c\cup\{dist(y,\Gamma\cup\Gamma_\infty)<2\delta\}} u\Box((1-\chi_2)\tilde{\chi}G)\, ds_y d\tau \\ &= u(t,x) + \int_{I^c\cup\{dist(y,\Gamma\cup\Gamma_\infty)<2\delta\}} u\Box((1-\chi_2)\tilde{\chi}G)\, ds_y d\tau\end{aligned}$$

and hence

$$0 = u(t,x) + \int_{I^c\cup\{dist(y,\Gamma\cup\Gamma_\infty)<2\delta\}} u\Box((1-\chi_2)\tilde{\chi}G)\, ds_y d\tau\,.$$

Concerning $\phi = \chi_2\tilde{\chi}G$, note that

$$
\begin{aligned}
0 &= \langle \Box(\chi_2\tilde{\chi}G), u\rangle \\
&= \int_{\mathbb{R}^+\times\Omega^e} \Box(\chi_2\tilde{\chi}G)u\, ds_y d\tau = \int_{\{dist(y,\Gamma\cup\Gamma_\infty)<2\delta\}} u\Box(\chi_2\tilde{\chi}G)\, ds_y d\tau \\
&\qquad + \int_{I\cap\{dist(y,\Gamma\cup\Gamma_\infty)>2\delta\}} u\Box(\chi_2\tilde{\chi}G)\, ds_y d\tau\,.
\end{aligned}
$$

Together we obtain

$$
\begin{aligned}
0 = u(t,x) &+ \int_{I^c\cup\{dist(y,\Gamma\cup\Gamma_\infty)<2\delta\}} u\Box((1-\chi_2)\tilde{\chi}G)\, ds_y d\tau \\
&+ \int_{\{dist(y,\Gamma\cup\Gamma_\infty)<2\delta\}} u\Box(\chi_2\tilde{\chi}G)\, ds_y d\tau + \int_{I\cap\{dist(y,\Gamma\cup\Gamma_\infty)>2\delta\}} u\Box(\chi_2\tilde{\chi}G)\, ds_y d\tau\,.
\end{aligned}
$$

Adding the formula for $(1-\tilde{\chi})G$ and letting δ go to 0 we obtain

$$
u(t,x) = -\langle \frac{\partial u}{\partial n}\delta_\Gamma + u\delta'_\Gamma + \frac{\partial u}{\partial n}\delta_{\Gamma_\infty} + u\delta'_{\Gamma_\infty}, G\rangle\,. \tag{3.12}
$$

<u>3. Step</u>: It remains to eliminate the contribution of Γ_∞. Using the boundary condition at Γ_∞ in (3.9), we see that

$$
\begin{aligned}
u(t,x) &= \int_{\mathbb{R}^+\times\Gamma} -\frac{\partial u}{\partial n}(\tau,y)G(t-\tau,x,y)ds_y d\tau + \int_{\mathbb{R}^+\times\Gamma} u(\tau,y)\frac{\partial G}{\partial n_y}(t-\tau,x,y)ds_y d\tau \\
&+ \int_{\mathbb{R}^+\times\Gamma_\infty} -\frac{\partial u}{\partial n}(\tau,y)G(t-\tau,x,y)ds_y d\tau + \int_{\mathbb{R}^+\times\Gamma_\infty} u(\tau,y)\frac{\partial G}{\partial n_y}(t-\tau,x,y)ds_y d\tau \\
&= \int_{\mathbb{R}^+\times\Gamma} -\frac{\partial u}{\partial n}(\tau,y)G(t-\tau,x,y)ds_y d\tau + \int_{\mathbb{R}^+\times\Gamma} u(\tau,y)\frac{\partial G}{\partial n_y}(t-\tau,x,y)ds_y d\tau \\
&+ \int_{\mathbb{R}^+\times\Gamma_\infty} \alpha_\infty\frac{\partial u}{\partial t}(\tau,y)G(t-\tau,x,y)ds_y d\tau + \int_{\mathbb{R}^+\times\Gamma_\infty} u(\tau,y)\frac{\partial G}{\partial n_y}(t-\tau,x,y)ds_y d\tau\,.
\end{aligned}
$$

Integration by parts in time yields

$$
\begin{aligned}
u(t,x) &= \int_{\mathbb{R}^+\times\Gamma} -\frac{\partial u}{\partial n}(\tau,y)G(t-\tau,x,y)ds_y d\tau + \int_{\mathbb{R}^+\times\Gamma} u(\tau,y)\frac{\partial G}{\partial n_y}(t-\tau,x,y)ds_y d\tau \\
&- \int_{\mathbb{R}^+\times\Gamma_\infty} \alpha_\infty\frac{\partial G}{\partial t}(t-\tau,x,y)u(\tau,y)ds_y d\tau + \int_{\mathbb{R}^+\times\Gamma_\infty} u(\tau,y)\frac{\partial G}{\partial n_y}(t-\tau,x,y)ds_y d\tau \\
&\quad + \int_{\Gamma_\infty} [\alpha_\infty u(\tau,y)G(t-\tau,x,y)]_0^\infty\, dx \\
&= \int_{\mathbb{R}^+\times\Gamma} -\frac{\partial u}{\partial n}(\tau,y)G(t-\tau,x,y)ds_y d\tau + \int_{\mathbb{R}^+\times\Gamma} u(\tau,y)\frac{\partial G}{\partial n_y}(t-\tau,x,y)ds_y d\tau\,.
\end{aligned}
$$

□

3.2.2 A System of BIEs for the Robin Problem

Associated Helmholtz Problem

As realized by Bamberger and Ha-Duong [6] [5], it is useful to analyse an associated Helmholtz problem in the frequency domain to obtain results for the wave equation.

We will therefore look at a scattering problem for acoustic waves in the half-space. For a fixed frequency ω with $\mathrm{Im}(\omega) > \sigma > 0$ the following exterior problem is considered:

$$\begin{cases} u^e \in H^1(\Omega^e), \\ (\Delta + \omega^2)u^e(x) = 0 \quad \text{in } \Omega^e \\ \frac{\partial u^e}{\partial n} + \alpha i\omega u^e = f \quad \text{on } \Gamma \\ \frac{\partial u^e}{\partial n} + \alpha_\infty i\omega u^e = 0 \quad \text{on } \Gamma_\infty \end{cases} \tag{3.13}$$

plus a decay condition(Sommerfeld's radiation condition) for the outgoing wave at infinity

$$u^e(x) = O(|x|^{-1}), \quad \frac{\partial u^e(x)}{\partial |x|} - i\omega u^e(x) = o(|x|^{-1}), \quad |x| \longrightarrow \infty\,,$$

(see e.g. Lemma 3.3 in [14] for the full space case and [25] for the impedance radiation condition in the half-space case). This condition holds automatically since for $\mathrm{Im}(\omega) > \sigma > 0$ the solution decays like $\exp(-\sigma|x|)$ and hence the solution belongs to H^1 and not only H^1_{loc}.
We also need an associated interior problem

$$\begin{cases} u^i \in H^1(\Omega^i), \\ (\Delta + \omega^2)u^i(x) = 0 \quad \text{in } \Omega^i \\ \frac{\partial u^i}{\partial n} - \alpha i\omega u^i = g \quad \text{on } \Gamma\,. \end{cases} \tag{3.14}$$

Here, α is the admittance function of the surface Γ and α_∞ is the admittance function of the planar surface Γ_∞. The right-hand sides f, g belong to $H^{-\frac{1}{2}}(\Gamma)$, and f is related to the incident wave by

$$f(x) = -\alpha(x) i\omega u_{inc}(x) - \frac{\partial u_{inc}}{\partial n}(x)\,.$$

The condition $u^e \in H^1(\Omega^e)$ replaces the Sommerfeld's radiation condition at infinity for real frequencies.

Theorem 3.2. *Let* $\mathrm{Im}(\omega) > 0$*. The problems* (3.13)-(3.14) *admit at most one solution for* $\mathrm{Re}(\alpha) \geq 0$ *and* $\mathrm{Re}(\alpha_\infty) \geq 0$.

Proof. We consider the homogeneous exterior problem

$$\begin{cases} u^e \in H^1(\Omega^e), \\ (\Delta + \omega^2)u^e(x) = 0 \quad \text{in } \Omega^e \\ \frac{\partial u^e}{\partial n} + \alpha i\omega u^e = 0 \quad \text{on } \Gamma \\ \frac{\partial u^e}{\partial n} + \alpha_\infty i\omega u^e = 0 \quad \text{on } \Gamma_\infty\,, \end{cases}$$

and the homogeneous interior problem

$$\begin{cases} u^i \in H^1(\Omega^i), \\ (\Delta + \omega^2)u^i(x) = 0 \quad \text{in } \Omega^i \\ \frac{\partial u^i}{\partial n} - \alpha i\omega u^i = 0 \quad \text{on } \Gamma\,. \end{cases}$$

We show that these problems admit at most one solution, namely zero. To do so we multiply the Helmholtz equation with $i\bar{\omega}\bar{u}$ and integrate over $\Omega^e \cup \Omega^i$. We obtain

$$\int_{\Omega^e \cup \Omega^i} \Delta u . i\bar{\omega}\bar{u} + \omega^2 u . i\bar{\omega}\bar{u}\, dx = 0 .$$

Now, we apply Green's first theorem to u^e and u^i to obtain

$$\int_{\Omega^e \cup \Omega^i} -i\bar{\omega}|\nabla u|^2 + i\omega|\omega|^2|u|^2\, dx - \int_\Gamma i\bar{\omega}(\frac{\partial u^e}{\partial n}\bar{u}^e - \frac{\partial u^i}{\partial n}\bar{u}^i)\, ds_x - \int_{\Gamma_\infty} i\bar{\omega}\frac{\partial u^e}{\partial n}\bar{u}^e\, ds_x = 0 .$$

Here, we have neglected a contribution from a large semi-ball which tends to zero as the radius of the semi-ball goes to infinity.

We take the real part of this equality and use the boundary conditions:

$$2\mathrm{Im}(\omega)\int_{\Omega^e \cup \Omega^i} |\nabla u|^2 + |\omega|^2|u|^2\, dx = \mathrm{Re}(\int_\Gamma -i\bar{\omega}(\frac{\partial u^e}{\partial n}\bar{u}^e - \frac{\partial u^i}{\partial n}\bar{u}^i)\, ds_x - \int_{\Gamma_\infty} i\bar{\omega}\frac{\partial u^e}{\partial n}\bar{u}^e\, ds_x)$$
$$= \mathrm{Re}(\int_\Gamma -i\bar{\omega}(-\alpha i\omega u^e\bar{u}^e - \alpha i\omega u^i\bar{u}^i)\, ds_x - i\bar{\omega}\int_{\Gamma_\infty} -\alpha_\infty i\omega u^e\bar{u}^e\, ds_x)$$
$$= \int_\Gamma -\mathrm{Re}(\alpha)(|\omega|^2|u^e|^2 + |\omega|^2|u^i|^2)\, ds_x + \int_{\Gamma_\infty} -\mathrm{Re}(\alpha_\infty)|\omega|^2|u^e|^2\, ds_x .$$

Since $\mathrm{Im}(\omega) > 0$, the conditions $\mathrm{Re}(\alpha) \geq 0$ and $\mathrm{Re}(\alpha_\infty) \geq 0$ ensure that $u^e = u^i = 0$ on $\Gamma_\infty \cup \Gamma$ and implies $u^e = u^i = 0$ everywhere on Ω^e resp. Ω^i. The uniqueness of the solution follows. □

The next step is to represent the solution of the wave equation in Ω^e and Ω^i by means of layer potentials using the representation formula for the Helmholtz equation.

A solution $u \in H^1(\Omega^i) \cup H^1(\Omega^e)$ of the Helmholtz equation can be expressed as:

$$u = S_\omega p - D_\omega \varphi \quad \text{in} \quad \Omega^i \cup \Omega^e ,$$

where

$$\varphi = u^i - u^e \quad \text{and} \quad p = \frac{\partial u^i}{\partial n} - \frac{\partial u^e}{\partial n} \quad \text{on } \Gamma .$$

S_ω is the single layer potential in the half-space associated to the absorbing boundary condition on Γ_∞ from (3.13):

$$S_\omega p(x) = \frac{1}{4\pi}\int_\Gamma \left(\frac{e^{i\omega|x-y|}}{4\pi|x-y|} + \frac{e^{i\omega|x-y'|}}{4\pi|x-y'|} \right.$$
$$\left. +2\beta_\infty e^{-\beta_\infty(x_3+y_3)}\int_\infty^{-(x_3+y_3)} e^{-\beta_\infty\eta}\frac{e^{ikr(\eta)}}{4\pi r(\eta)}d\eta \right) p(y)\, ds_y$$

where $r(\eta) = \sqrt{\varrho_s^2 + \eta^2}$, $\varrho_s = |x_s - y_s| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$ and $\beta_\infty = i\omega\alpha_\infty$. D_ω is the corresponding double layer potential

$$D_\omega\varphi(x) = \frac{1}{4\pi}\int_\Gamma \frac{\partial}{\partial n_y}\left(\frac{e^{i\omega|x-y|}}{4\pi|x-y|} + \frac{e^{i\omega|x-y'|}}{4\pi|x-y'|} \right.$$
$$\left. +2\beta_\infty e^{-\beta_\infty(x_3+y_3)}\int_\infty^{-(x_3+y_3)} e^{-\beta_\infty\eta}\frac{e^{ikr(\eta)}}{4\pi r(\eta)}d\eta \right) p(y)\, ds_y .$$

The relevant integral operators $V_\omega,\ K_\omega,\ K'_\omega,\ W_\omega$ on Γ are:

$$V_\omega p(x) = \frac{1}{2\pi}\int_\Gamma \left(\frac{e^{i\omega|x-y|}}{4\pi|x-y|} + \frac{e^{i\omega|x-y'|}}{4\pi|x-y'|} \right.$$
$$\left. +2\beta_\infty e^{-\beta_\infty(x_3+y_3)} \int_\infty^{-(x_3+y_3)} e^{-\beta_\infty\eta} \frac{e^{ikr(\eta)}}{4\pi r(\eta)} d\eta \right) p(y)\, ds_y$$

$$K'_\omega \varphi(x) = \frac{1}{2\pi}\int_\Gamma \frac{\partial}{\partial n_x} \left(\frac{e^{i\omega|x-y|}}{4\pi|x-y|} + \frac{e^{i\omega|x-y'|}}{4\pi|x-y'|} \right.$$
$$\left. +2\beta_\infty e^{-\beta_\infty(x_3+y_3)} \int_\infty^{-(x_3+y_3)} e^{-\beta_\infty\eta} \frac{e^{ikr(\eta)}}{4\pi r(\eta)} d\eta \right) p(y)\, ds_y$$

$$K_\omega \varphi(x) = \frac{1}{2\pi}\int_\Gamma \frac{\partial}{\partial n_y} \left(\frac{e^{i\omega|x-y|}}{4\pi|x-y|} + \frac{e^{i\omega|x-y'|}}{4\pi|x-y'|} \right.$$
$$\left. +2\beta_\infty e^{-\beta_\infty(x_3+y_3)} \int_\infty^{-(x_3+y_3)} e^{-\beta_\infty\eta} \frac{e^{ikr(\eta)}}{4\pi r(\eta)} d\eta \right) p(y)\, ds_y$$

$$W_\omega \varphi(x) = \frac{1}{2\pi}\int_\Gamma \frac{\partial^2}{\partial n_x n_y} \left(\frac{e^{i\omega|x-y|}}{4\pi|x-y|} + \frac{e^{i\omega|x-y'|}}{4\pi|x-y'|} \right.$$
$$\left. +2\beta_\infty e^{-\beta_\infty(x_3+y_3)} \int_\infty^{-(x_3+y_3)} e^{-\beta_\infty\eta} \frac{e^{ikr(\eta)}}{4\pi r(\eta)} d\eta \right) p(y)\, ds_y\,.$$

We use them to express the traces of u in term of φ and p:

$$\begin{aligned} 2u^e &= V_\omega p - (I + K_\omega)\varphi \\ 2u^i &= V_\omega p + (I - K_\omega)\varphi \\ 2\frac{\partial u^e}{\partial n} &= (-I + K'_\omega)p - W_\omega\varphi \\ 2\frac{\partial u^i}{\partial n} &= (I + K'_\omega)p - W_\omega\varphi\,. \end{aligned} \tag{3.15}$$

Adding and substracting the boundary conditions (3.13)-(3.14) on Γ, we have

$$\begin{cases} \frac{\partial u^e}{\partial n} + \frac{\partial u^i}{\partial n} - \alpha i\omega\varphi = f + g = F \\ p - \alpha i\omega(u^e + u^i) = g - f = G\,. \end{cases}$$

Then using the equation (3.15) of the trace u we find the following system of integral equations:

$$\begin{cases} (K'_\omega p - W_\omega\varphi) - i\omega\alpha\varphi = F \\ p - i\omega\alpha(V_\omega p - K_\omega\varphi) = G\,. \end{cases} \tag{3.16}$$

If $\alpha \neq 0$ multiplying the first equation by $\overline{-i\omega\psi}$ and the second by $\frac{1}{\alpha}\bar{q}$, after integration we obtain the weak formulation:

$$a_\omega(\tilde{U}, \tilde{V}) = l_\omega(\tilde{V}). \tag{3.17}$$

Here,

$$\begin{aligned} a_\omega(\tilde{U}, \tilde{V}) = |\omega|^2 \int_\Gamma \alpha\varphi\bar{\psi} ds_x + \int_\Gamma \frac{1}{\alpha} p\bar{q} ds_x + i\bar{\omega} \int_\Gamma K'_\omega p\bar{\psi} ds_x \\ -i\bar{\omega} \int_\Gamma W_\omega \varphi\bar{\psi} ds_x - i\omega \int_\Gamma V_\omega p\bar{q} ds_x + i\omega \int_\Gamma K_\omega \varphi\bar{q} ds_x \end{aligned}$$

and

$$l_\omega(\tilde{V}) = i\bar{\omega} \int_\Gamma F\bar{\psi} ds_x + \int_\Gamma \frac{1}{\alpha} G\bar{q} ds_x \,,$$
$$\tilde{U} = (\varphi, p) \,, \tilde{V} = (\psi, q) \,.$$

For $\alpha = 0$, (3.16) reduces to $W_\omega \varphi = K_\omega G - F$. The analysis of this case is analogous.

Theorem 3.3. (Coercivity)
Assume that $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\alpha_\infty) \geq 0$. *Then the following inequality holds for all* $\tilde{U} = (\varphi, p) \in H^{\frac{1}{2}}(\Gamma) \times L^2(\Gamma)$:

$$\mathrm{Re}(a_\omega(\tilde{U}, \tilde{U})) \geq ||\sqrt{\mathrm{Re}(\frac{1}{\alpha})} p||^2_{0,\Gamma} + C_\sigma ||\varphi||^2_{\frac{1}{2},\omega,\Gamma} + ||\omega\sqrt{\mathrm{Re}(\alpha)}\varphi||^2_{0,\Gamma} \,.$$

Proof. Taking the real part of the bilinear form a_ω and using (3.15), we calculate

$$\begin{aligned} \mathrm{Re}(a_\omega(\tilde{U}, \tilde{U})) &= \mathrm{Re} \int_\Gamma (K'_\omega p - W_\omega \varphi - i\omega\alpha\varphi)(\overline{-i\omega\varphi}) + \bar{p} \frac{p - i\omega\alpha(V_\omega p - K_\omega \varphi)}{\alpha} \, ds_x \\ &= \mathrm{Re} \int_\Gamma [\frac{\partial u^i}{\partial n} + \frac{\partial u^e}{\partial n} - i\omega\alpha(u^i - u^e)] i\bar{\omega}(\bar{u^i} - \bar{u^e}) \, ds_x \\ &\quad + \mathrm{Re} \int_\Gamma \frac{1}{\alpha} (\frac{\partial \bar{u^i}}{\partial n} - \frac{\partial \bar{u^e}}{\partial n})(\frac{\partial u^i}{\partial n} - \frac{\partial u^e}{\partial n} - i\omega\alpha(u^i + u^e)) \, ds_x \\ &= \mathrm{Re} \int_\Gamma i\bar{\omega}(2\frac{\partial u^i}{\partial n}\bar{u^i} - 2\frac{\partial u^e}{\partial n}\bar{u^e}) \, ds_x \\ &\quad + \int_\Gamma \frac{1}{\alpha} \underbrace{|\frac{\partial \bar{u^i}}{\partial n} - \frac{\partial \bar{u^e}}{\partial n}|^2}_{=|p|^2} \, ds_x + |\omega|^2 \int_\Gamma \alpha \underbrace{|u^i - u^e|^2}_{=|\varphi|^2} \, ds_x \,. \end{aligned}$$

By Green's formula using the derivative in direction of the interior normal.

$$\begin{aligned} -\int_\Gamma \frac{\partial u^e}{\partial n} \bar{u^e} \, ds_x &= -\int_\Gamma (\text{interior normal derivative on } \Gamma \text{ of } u^e)\bar{u^e} \\ &= -\int_\Gamma \frac{\partial u^e}{\partial n} \bar{u^e} \, ds_x - \int_{\Gamma_\infty} \frac{\partial u^e}{\partial n} \bar{u^e} \, ds_x + \int_{\Gamma_\infty} \underbrace{\frac{\partial u^e}{\partial n}}_{\frac{\partial u^e}{\partial x_3}} \bar{u^e} \, ds_x \,. \end{aligned}$$

Integration by parts on Ω^e leads to:

$$
\begin{aligned}
-\int_\Gamma \frac{\partial u^e}{\partial n}\bar{u^e}\,ds_x &= \int_{\Omega^e} \Delta u^e\,\bar{u^e} + \nabla u^e\,\overline{\nabla u^e}\,dx + \int_{\Gamma_\infty} \frac{\partial u^e}{\partial x_3}\bar{u^e}\,ds_x \\
&= \int_{\Omega^e} |\nabla u^e|^2 - \omega^2|u^e|^2\,dx + \int_{\Gamma_\infty} \frac{\partial u^e}{\partial x_3}\bar{u^e}\,ds_x \\
&= \int_{\Omega^e} |\nabla u^e|^2 - \omega^2|u^e|^2\,dx - \int_{\Gamma_\infty} i\alpha_\infty\omega|u^e|^2\,ds_x\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
-\mathrm{Re}\,2i\bar\omega\int_\Gamma \frac{\partial u^e}{\partial n}\bar{u^e}\,ds_x &= \mathrm{Re}(2\int_{\Omega^e} i\bar\omega|\nabla u^e|^2 - i\bar\omega\omega^2|u^e|^2\,dx - 2\int_{\Gamma_\infty}(i\bar\omega)i\alpha_\infty\omega|u^e|^2\,ds_x) \\
&= \mathrm{Re}(2\int_{\Omega^e} i\bar\omega|\nabla u^e|^2 - i\omega|\omega|^2|u^e|^2\,dx + 2\int_{\Gamma_\infty}\alpha_\infty|\omega|^2|u^e|^2\,ds_x) \\
&= 2\sigma\int_{\Omega^e}|\nabla u^e|^2 + |\omega|^2|u^e|^2\,dx + 2\int_{\Gamma_\infty}\alpha_\infty|\omega|^2|u^e|^2\,ds_x\,.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\mathrm{Re}\,2i\bar\omega\int_\Gamma \frac{\partial u^i}{\partial n}\bar{u^i}\,ds_x &= \mathrm{Re}(2\int_{\Omega^i} i\bar\omega|\nabla u^i|^2 - i\bar\omega\omega^2|u^i|^2\,dx) \\
&= 2\sigma\int_{\Omega^i}|\nabla u^i|^2 + |\omega|^2|u^i|^2\,dx\,.
\end{aligned}
$$

We conclude:

$$
\begin{aligned}
\mathrm{Re}(a_\omega(\tilde U,\tilde U)) &= \mathrm{Re}\,2i\bar\omega\int_\Gamma(\frac{\partial u^i}{\partial n}\bar{u^i} - \frac{\partial u^e}{\partial n}\bar{u^e})\,ds_x + \int_\Gamma \frac{1}{\alpha}|p|^2 + |\omega|^2\int_\Gamma \alpha|\varphi|^2\,ds_x \\
&= 2\sigma\int_{\Omega^i\cup\Omega^e}|\nabla u|^2 + |\omega|^2|u|^2\,dx + \int_\Gamma \mathrm{Re}(\frac{1}{\alpha})|p|^2\,ds_x + |\omega|^2\int_\Gamma \mathrm{Re}(\alpha)|\varphi|^2\,ds_x \\
&\qquad + 2\int_{\Gamma_\infty}\mathrm{Re}(\alpha_\infty)|\omega|^2|u^e|^2\,ds_x \\
&\geq 2\sigma\int_{\Omega^i\cup\Omega^e}|\nabla u|^2 + |\omega|^2|u|^2\,dx + \int_\Gamma \mathrm{Re}(\frac{1}{\alpha})|p|^2\,ds_x + |\omega|^2\int_\Gamma \mathrm{Re}(\alpha)|\varphi|^2\,ds_x\,.
\end{aligned}
$$

Using the trace theorem in Ω^i and Ω^e

$$
||\varphi||_{\frac{1}{2},\omega,\Gamma} \leq C||u||_{1,\omega,\Omega}\,,
$$

it follows that

$$
\mathrm{Re}(a_\omega(\tilde U,\tilde U)) \geq ||\sqrt{\mathrm{Re}(\frac{1}{\alpha})}p||^2_{0,\Gamma} + C_\sigma||\varphi||^2_{\frac{1}{2},\omega,\Gamma} + ||\omega\sqrt{\mathrm{Re}(\alpha)}\varphi||^2_{0,\Gamma}\,.
$$

□

Remark 3.3. *Assume* $\mathrm{Re}(\alpha_\infty) \geq 0$. *Then a similar coercivity estimate holds for the single layer potential* V_ω*:*

$$
\mathrm{Re}\langle i\omega V_\omega\dot\varphi,\varphi\rangle \geq C_\sigma||\varphi||^2_{-\frac{1}{2},\omega,\Gamma}\,. \tag{3.18}
$$

Theorem 3.4. (Continuity)
Assume that $\mathrm{Re}(\alpha_\infty) \geq 0$. *The integral operators satisfy the following mapping properties for* $p \in H^{-\frac{1}{2}}(\Gamma)$ *and* $\varphi \in H^{\frac{1}{2}}(\Gamma)$*:*

$$||V_\omega p||_{\frac{1}{2},\omega,\Gamma} \leq C_\sigma |\omega| ||p||_{-\frac{1}{2},\omega,\Gamma}\,, \tag{3.19}$$

$$||W_\omega \varphi||_{-\frac{1}{2},\omega,\Gamma} \leq C_\sigma |\omega| ||\varphi||_{\frac{1}{2},\omega,\Gamma}\,, \tag{3.20}$$

$$||(I - K_\omega)\varphi||_{\frac{1}{2},\omega,\Gamma} \leq C_\sigma |\omega| ||\varphi||_{\frac{1}{2},\omega,\Gamma}\,, \tag{3.21}$$

$$||(I - K'_\omega)p||_{-\frac{1}{2},\omega,\Gamma} \leq C_\sigma |\omega| ||p||_{-\frac{1}{2},\omega,\Gamma}\,. \tag{3.22}$$

Proof. First we prove (3.19).
Let be p in $H^{-\frac{1}{2}}(\Gamma)$ and let $v = S_\omega \varphi$. Then we saw that v verifies:

$$\begin{cases} (\Delta + \omega^2)v(x) = 0 \quad \text{ in } \Omega^i \cup \Omega^e \\ \frac{\partial v^i}{\partial n} - \frac{\partial v^e}{\partial n} = p \quad \text{ on } \Gamma \\ v^i - v^e = 0 \quad \text{ on } \Gamma\,. \end{cases}$$

Applying Green's Theorem in $\Omega^i \cup \Omega^e$ we obtain

$$\int_\Gamma i\bar{\omega} \frac{\partial v^i}{\partial n} \bar{v^i}\, ds_x = \int_{\Omega^i} -i\omega|\omega|^2 v^i \bar{v^i}\, dx + \int_{\Omega^i} i\bar{\omega} \nabla v^i \nabla \bar{v^i}\, dx \tag{3.23}$$

and

$$-\int_{\Gamma_\infty} i\bar{\omega} \frac{\partial v^e}{\partial x_3} \bar{v^e}\, ds_x - \int_\Gamma i\bar{\omega} \frac{\partial v^e}{\partial n} \bar{v^e}\, ds_x = \int_{\Omega^e} -i\omega|\omega|^2 v^e \bar{v^e}\, dx + \int_{\Omega^e} i\bar{\omega} \nabla v^e \nabla \bar{v^e}\, dx\,. \tag{3.24}$$

Adding the two equations (3.23) and (3.24) we get

$$-\int_{\Gamma_\infty} i\bar{\omega} \frac{\partial v^e}{\partial x_3} \bar{v^e}\, ds_x + \int_\Gamma i\bar{\omega} \left(\frac{\partial v^i}{\partial n} - \frac{\partial v^e}{\partial n}\right) \bar{v^e}\, ds_x = \int_{\Omega^e \cup \Omega^i} -i\omega|\omega|^2 |v|^2\, dx + \int_{\Omega^e \cup \Omega^i} i\bar{\omega} |\nabla v|^2\, dx.$$

Using the boundary conditions on Γ and Γ_∞ we obtain

$$-\int_{\Gamma_\infty} \alpha_\infty |\omega|^2 |v^e|^2\, ds_x + \int_\Gamma i\bar{\omega} p \bar{v^e}\, ds_x = \int_{\Omega^e \cup \Omega^i} -i\omega|\omega|^2 |v|^2\, dx + \int_{\Omega^e \cup \Omega^i} i\bar{\omega} |\nabla v|^2\, dx\,.$$

We take the real part of this equation:

$$-\int_{\Gamma_\infty} \mathrm{Re}(\alpha_\infty) |\omega|^2 |v^e|^2\, ds_x + \mathrm{Re}\left(\int_\Gamma i\bar{\omega} p \bar{v^e}\, ds_x\right) = \sigma\left(||v^e||^2_{1,\omega,\Omega^e} + ||v^i||^2_{1,\omega,\Omega^i}\right)\,.$$

It follows from $\mathrm{Re}(\alpha_\infty) \geq 0$ and from the trace theorem (Lemma 1.4) that

$$\mathrm{Re}\left(\int_\Gamma i\bar{\omega} p \bar{v^e}\, ds_x\right) \geq \sigma\left(||v^e||^2_{\frac{1}{2},\omega,\Gamma} + ||v^i||^2_{\frac{1}{2},\omega,\Gamma}\right)\,.$$

Therefore :

$$|\omega|||p||_{-\frac{1}{2},\omega,\Gamma}||v^e||_{\frac{1}{2},\omega,\Gamma} \geq 2\sigma||v^e||^2_{\frac{1}{2},\omega,\Gamma}$$
$$|\omega|||p||_{-\frac{1}{2},\omega,\Gamma} \geq 2\sigma||v^e||_{\frac{1}{2},\omega,\Gamma}$$

and $v_{|\Gamma} = V_\omega p$ imply that

$$||V_\omega p||_{\frac{1}{2},\omega,\Gamma} \leq \frac{|\omega|}{2\sigma}||p||_{-\frac{1}{2},\omega,\Gamma}\,.$$

Now we prove the estimate (3.20). Let φ in $H^{\frac{1}{2}}(\Gamma)$ and let $v = -D_\omega\varphi$. Then we saw that v verifies:

$$\begin{cases} (\Delta+\omega^2)v(x) = 0 \quad \text{ in } \Omega^i \cup \Omega^e \\ \frac{\partial v^i}{\partial n} - \frac{\partial v^e}{\partial n} = 0 \quad \text{ on } \Gamma \\ v^i - v^e = \varphi \quad \text{ on } \Gamma\,. \end{cases}$$

Moreover we have $\frac{\partial v^e}{\partial n} = -W_\omega\varphi$.
Adding the two equations (3.23) and (3.24) we obtain

$$-\int_{\Gamma_\infty} i\bar{\omega}\frac{\partial v^e}{\partial x_3}\bar{v^e} + \int_\Gamma i\bar{\omega}\frac{\partial v^e}{\partial n}\left(\bar{v^i} - \bar{v^e}\right) = \int_{\Omega^e\cup\Omega^i} -i\omega|\omega|^2|v|^2 + \int_{\Omega^e\cup\Omega^i} i\bar{\omega}|\nabla v|^2\,.$$

By using the boundary condition on Γ and Γ_∞ we obtain the following equality:

$$-\int_{\Gamma_\infty} \alpha_\infty|\omega|^2|v^e|^2\,ds_x + \int_\Gamma i\bar{\omega}\frac{\partial v^e}{\partial n}\bar{\varphi}\,ds_x = \int_{\Omega^e\cup\Omega^i} -i\omega|\omega|^2|v|^2\,dx + \int_{\Omega^e\cup\Omega^i} i\bar{\omega}|\nabla v|^2\,dx\,.$$

We take the real part of this equation:

$$-\int_{\Gamma_\infty} \mathrm{Re}(\alpha_\infty)|\omega|^2|v^e|^2\,ds_x + \mathrm{Re}\left(\int_\Gamma i\bar{\omega}\frac{\partial v^e}{\partial n}\bar{\varphi}\,ds_x\right) = \sigma\left(||v^e||^2_{1,\omega,\Omega^e} + ||v^i||^2_{1,\omega,\Omega^i}\right)\,.$$

Because $\mathrm{Re}(\alpha_\infty) \geq 0$ and using Cauchy-Schwarz we obtain

$$||v^e||^2_{1,\omega,\Omega^e} \leq \frac{1}{\mathrm{Im}(\omega)}|\omega|||\varphi||_{\frac{1}{2},\omega,\Gamma}||\frac{\partial v^e}{\partial n}||_{-\frac{1}{2},\omega,\Gamma}\,. \tag{3.25}$$

Using Green's theorem in Ω^e we obtain

$$-\int_{\Gamma_\infty}\frac{\partial v^e}{\partial x_3}\bar{\varphi}\,ds_x - \int_\Gamma\frac{\partial v^e}{\partial n}\bar{\varphi}\,ds_x = \int_{\Omega^e} -\omega^2 v^e\bar{\psi}\,dx + \int_{\Omega^e}\nabla v^e\nabla\bar{\psi}\,dx\,,$$

where $\quad\psi \in H^1(\Omega^e)$ as Lemma 1.3 and $\quad\varphi = \psi_{|\Gamma} \in H^{\frac{1}{2}}(\Gamma)$.
For $\varphi \in H^{\frac{1}{2}}(\Gamma)$ we have by the trace theorem

$$|\int_\Gamma\frac{\partial v^e}{\partial n}\bar{\varphi}\,ds_x| \leq ||v^e||_{1,\omega,\Omega^e}||\psi||_{1,\omega,\Omega^e}\,.$$

By Lemma 1.3

$$||\psi||_{1,\omega,\Omega^e} \leq C||\phi||_{\frac{1}{2},\omega,\Gamma}\,.$$

It follows that

$$||\frac{\partial v^e}{\partial n}||_{-\frac{1}{2},\omega,\Gamma} = sup_{\{\phi \in H^{\frac{1}{2}}(\Gamma)/||\phi||_{\frac{1}{2},\omega,\Gamma}=1\}} |\int_\Gamma \frac{\partial v^e}{\partial n} \bar{\phi}\, ds_x| \leq C||v^e||_{1,\omega,\Omega^e} .$$

From (3.25) it follows that

$$||W_\omega \varphi||_{-\frac{1}{2},\omega,\Gamma} = ||\frac{\partial v^e}{\partial n}||_{-\frac{1}{2},\omega,\Gamma} \leq C||v^e||_{1,\omega,\Omega^e} \leq C|\omega|||\varphi||_{\frac{1}{2},\omega,\Gamma} .$$

Using similar reasoning, we obtain the estimates (3.22) and (3.21).

□

Theorem 3.5. (Continuity)
Assume that $\mathrm{Re}(\alpha_\infty) \geq 0$ *and* α, $\frac{1}{\alpha} \in L^\infty(\Gamma)$. *The bilinear form* a_ω *is continuous on* $\left(H^{\frac{1}{2}}(\Gamma) \times L^2(\Gamma)\right) \times \left(H^{\frac{1}{2}}(\Gamma) \times L^2(\Gamma)\right)$.

Proof. Recall that with $\tilde{U} = (\varphi, p)$ and $\tilde{V} = (\psi, q)$

$$\begin{aligned} a_\omega(\tilde{U}, \tilde{V}) = |\omega|^2 \int_\Gamma \alpha\varphi\bar{\psi} ds_x + \int_\Gamma \frac{1}{\alpha} p\bar{q} ds_x + i\bar{\omega} \int_\Gamma K'_\omega p\bar{\psi} ds_x \\ -i\bar{\omega} \int_\Gamma W_\omega \varphi\bar{\psi} ds_x - i\omega \int_\Gamma V_\omega p\bar{q} ds_x + i\omega \int_\Gamma K_\omega \varphi\bar{q} ds_x . \end{aligned}$$

Now we estimate the various terms of the bilinear form a_ω using Theorem 3.4

$$|\omega|^2 \left| \int_\Gamma \alpha\varphi\bar{\psi} ds_x \right| \leq C|\omega|^2||\varphi||_{0,\omega,\Gamma}||\psi||_{0,\omega,\Gamma} \leq C|\omega|^2||\varphi||_{\frac{1}{2},\omega,\Gamma}||\psi||_{\frac{1}{2},\omega,\Gamma} ,$$

$$\left| \int_\Gamma \frac{1}{\alpha} p\bar{q} ds_x \right| \leq C||p||_{0,\omega,\Gamma}||q||_{0,\omega,\Gamma} \leq \frac{C}{\sigma^2}|\omega|^2||p||_{0,\omega,\Gamma}||q||_{0,\omega,\Gamma} ,$$

$$\begin{aligned} \left| i\bar{\omega} \int_\Gamma K'_\omega p\bar{\psi} ds_x \right| &\leq \frac{|\omega|}{\sigma}||K'_\omega p||_{-\frac{1}{2},\omega,\Gamma}||\psi||_{\frac{1}{2},\omega,\Gamma} \\ &\leq C\frac{|\omega|^2}{\sigma}||p||_{-\frac{1}{2},\omega,\Gamma}||\psi||_{\frac{1}{2},\omega,\Gamma} \\ &\leq C\frac{|\omega|^2}{\sigma}||p||_{0,\omega,\Gamma}||\psi||_{\frac{1}{2},\omega,\Gamma} , \end{aligned}$$

$$\left| i\bar{\omega} \int_\Gamma W_\omega \varphi\bar{\psi} ds_x \right| \leq |\omega|||W_\omega \varphi||_{-\frac{1}{2},\omega,\Gamma}||\psi||_{\frac{1}{2},\omega,\Gamma} \leq C|\omega|^2||\varphi||_{\frac{1}{2},\omega,\Gamma}||\psi||_{\frac{1}{2},\omega,\Gamma} ,$$

$$\begin{aligned} \left| i\omega \int_\Gamma V_\omega p\bar{q} ds_x \right| &\leq |\omega|||V_\omega p||_{\frac{1}{2},\omega,\Gamma}||q||_{-\frac{1}{2},\omega,\Gamma} \\ &\leq C|\omega|^2||p||_{-\frac{1}{2},\omega,\Gamma}||q||_{-\frac{1}{2},\omega,\Gamma} \\ &\leq C|\omega|^2||p||_{0,\omega,\Gamma}||q||_{0,\omega,\Gamma} \end{aligned}$$

$$\begin{aligned}\left| i\omega \int_\Gamma K_\omega \varphi \bar{q} ds_x \right| &\leq |\omega| ||K_\omega \varphi||_{\frac{1}{2},\omega,\Gamma} ||q||_{-\frac{1}{2},\omega,\Gamma} \\ &\leq C|\omega|^2 ||\varphi||_{\frac{1}{2},\omega,\Gamma} ||q||_{-\frac{1}{2},\omega,\Gamma} \\ &\leq C|\omega|^2 ||\varphi||_{\frac{1}{2},\omega,\Gamma} ||q||_{0,\omega,\Gamma}\end{aligned}$$

Adding the 6 inequalities we get

$$a_\omega(\tilde{U}, \tilde{V}) \leq C_\sigma \left(|\omega||p|_{0,\omega,\Gamma} + |\omega||\varphi|_{\frac{1}{2},\omega,\Gamma} \right) \left(|\omega||q|_{0,\omega,\Gamma} + |\omega||\psi|_{\frac{1}{2},\omega,\Gamma} \right) . \tag{3.26}$$

□

Time-dependent Problem

We consider a time-dependent acoustic scattering problem with an absorbing object in the half-space above the absorbing plane Γ_∞.
The scattered wave then satisfies the following initial boundary value problem:

$$\begin{cases} (\frac{\partial^2}{\partial t^2} - \Delta) u^e(t,x) = 0 \quad \text{ in } \mathbb{R}^+ \times \Omega^e \\ \frac{\partial u^e}{\partial n} - \alpha \frac{\partial u^e}{\partial t} = f \quad \text{ on } \mathbb{R}^+ \times \Gamma \\ \frac{\partial u^e}{\partial n} - \alpha_\infty \frac{\partial u^e}{\partial t} = 0 \quad \text{ on } \mathbb{R}^+ \times \Gamma_\infty \\ \frac{\partial u^e}{\partial t}(0,x) = u^e(0,x) = 0 \quad \text{ in } \Omega^e . \end{cases} \tag{3.27}$$

The corresponding interior problem is:

$$\begin{cases} (\frac{\partial^2}{\partial t^2} - \Delta) u^i(t,x) = 0 \quad \text{ in } \mathbb{R}^+ \times \Omega^i \\ \frac{\partial u^i}{\partial n} + \alpha \frac{\partial u^i}{\partial t} = g \quad \text{ on } \mathbb{R}^+ \times \Gamma \\ \frac{\partial u^i}{\partial t}(0,x) = u^i(0,x) = 0 \quad \text{ in } \Omega^i . \end{cases} \tag{3.28}$$

The solution of (3.27) and (3.28) satisfies the representation formula

$$u(t,x) = Sp(t,x) - D\varphi(t,x) \quad \forall (t,x) \in \mathbb{R}^+ \times \Omega^e \cup \Omega^i ,$$

where

$$\varphi = u^i - u^e , \quad p = \frac{\partial u^i}{\partial n} - \frac{\partial u^e}{\partial n} \quad \text{ on } \mathbb{R}^+ \times \Gamma .$$

S is the single layer potential in the time domain for a half-space with an absorbing boundary condition, defined by

$$\begin{aligned} Sp(t,x) = & \frac{1}{4\pi} \int_\Gamma \frac{p(t - |x-y|, y)}{|x-y|} ds_y + \frac{1}{4\pi} \int_\Gamma \frac{p(t - |x-y'|, y)}{|x-y'|} ds_y \\ & - \frac{\alpha_\infty}{2\pi} \int_0^\infty \int_\Gamma \frac{\partial}{\partial s} [\frac{H(t-s-|x-y'|)}{\sqrt{(t-s+\alpha_\infty \vartheta_3)^2 + (\alpha_\infty^2 - 1)R^2}}] p(s,y) ds_y ds . \end{aligned}$$

The corresponding double layer potential D is:

$$
\begin{aligned}
D\varphi(t,x) =& \frac{1}{4\pi}\int_\Gamma \frac{\partial}{\partial n_y}\left(\frac{p(t-|x-y|,y)}{|x-y|}ds_y\right) + \frac{1}{4\pi}\int_\Gamma \frac{\partial}{\partial n_y}\left(\frac{p(t-|x-y'|,y)}{|x-y'|}\right) ds_y \\
&- \frac{\alpha_\infty}{2\pi}\int\limits_0^\infty \int_\Gamma \frac{\partial}{\partial n_y}\left(\frac{\partial}{\partial s}[\frac{H(t-s-|x-y'|)}{\sqrt{(t-s+\alpha_\infty\vartheta_3)^2+(\alpha_\infty^2-1)R^2}}]p(s,y)\right) ds_y ds\,.
\end{aligned}
$$

As for the Helmholtz equation we have the following trace identities:

$$
\begin{aligned}
2u^e &= Vp - (I+K)\varphi \\
2u^i &= Vp + (I-K)\varphi \\
2\frac{\partial u^e}{\partial n} &= (-I+K')p - W\varphi \\
2\frac{\partial u^i}{\partial n} &= (I+K')p - W\varphi\,.
\end{aligned}
\tag{3.29}
$$

The relevant boundary integral operators on Γ are:

$$
\begin{aligned}
Vp(t,x) =& \frac{1}{2\pi}\int_\Gamma \frac{p(t-|x-y|,y)}{|x-y|}ds_y + \frac{1}{2\pi}\int_\Gamma \frac{p(t-|x-y'|,y)}{|x-y'|}ds_y \\
&- \frac{\alpha_\infty}{\pi}\int\limits_0^\infty \int_\Gamma \frac{\partial}{\partial s}[\frac{H(t-s-|x-y'|)}{\sqrt{(t-s+\alpha_\infty\vartheta_3)^2+(\alpha_\infty^2-1)R^2}}]p(s,y) ds_y ds\,,
\end{aligned}
$$

$$
\begin{aligned}
K'\varphi(t,x) =& \frac{1}{2\pi}\int_\Gamma \frac{\partial}{\partial n_x}\left(\frac{p(t-|x-y|,y)}{|x-y|}ds_y\right) + \frac{1}{2\pi}\int_\Gamma \frac{\partial}{\partial n_x}\left(\frac{p(t-|x-y'|,y)}{|x-y'|}\right) ds_y \\
&- \frac{\alpha_\infty}{\pi}\int\limits_0^\infty \int_\Gamma \frac{\partial}{\partial n_x}\left(\frac{\partial}{\partial s}[\frac{H(t-s-|x-y'|)}{\sqrt{(t-s+\alpha_\infty\vartheta_3)^2+(\alpha_\infty^2-1)R^2}}]p(s,y)\right) ds_y ds\,,
\end{aligned}
$$

$$
\begin{aligned}
K\varphi(t,x) =& \frac{1}{2\pi}\int_\Gamma \frac{\partial}{\partial n_y}\left(\frac{p(t-|x-y|,y)}{|x-y|}ds_y\right) + \frac{1}{2\pi}\int_\Gamma \frac{\partial}{\partial n_y}\left(\frac{p(t-|x-y'|,y)}{|x-y'|}\right) ds_y \\
&- \frac{\alpha_\infty}{\pi}\int\limits_0^\infty \int_\Gamma \frac{\partial}{\partial n_y}\left(\frac{\partial}{\partial s}[\frac{H(t-s-|x-y'|)}{\sqrt{(t-s+\alpha_\infty\vartheta_3)^2+(\alpha_\infty^2-1)R^2}}]p(s,y)\right) ds_y ds\,,
\end{aligned}
$$

$$
\begin{aligned}
W\varphi(t,x) =& \frac{1}{2\pi}\int_\Gamma \frac{\partial^2}{\partial n_x n_y}\left(\frac{p(t-|x-y|,y)}{|x-y|}ds_y\right) + \frac{1}{2\pi}\int_\Gamma \frac{\partial^2}{\partial n_x n_y}\left(\frac{p(t-|x-y'|,y)}{|x-y'|}\right) ds_y \\
&- \frac{\alpha_\infty}{\pi}\int\limits_0^\infty \int_\Gamma \frac{\partial^2}{\partial n_x n_y}\left(\frac{\partial}{\partial s}[\frac{H(t-s-|x-y'|)}{\sqrt{(t-s+\alpha_\infty\vartheta_3)^2+(\alpha_\infty^2-1)R^2}}]p(s,y)\right) ds_y ds\,.
\end{aligned}
$$

Here $\vartheta_3 = x_3 + y_3$ and $R^2 = (x_1-y_1)^2 + (x_2-y_2)^2$.
Substituting formula (3.29) into the boundary condition on Γ, we obtain the following

system for the unknown functions φ and p

$$\begin{cases}(-I+K')p - W\varphi - \alpha\frac{\partial}{\partial t}(Vp-(I+K)\varphi) = 2f \\ (I+K')p - W\varphi + \alpha\frac{\partial}{\partial t}(Vp+(I-K)\varphi) = 2g\,. \end{cases} \tag{3.30}$$

Adding respectively substracting the two equations of (3.30), leads to

$$\begin{cases}(K'p - W\varphi) + \alpha\frac{\partial\varphi}{\partial t} = F \\ p + \alpha(V\frac{\partial p}{\partial t} - K\frac{\partial\varphi}{\partial t}) = G\,. \end{cases} \tag{3.31}$$

Pairing these equations with test functions $\dot{\psi}$ respectively $\frac{q}{\alpha}$, we obtain the following space-time variational formulation:

$$\int_0^\infty e^{-2\sigma t}\int_\Gamma \left[(K'p - W\varphi) + \alpha\dot{\varphi}\right]\dot{\psi}\,ds_x\,dt = \int_0^\infty e^{-2\sigma t}\int_\Gamma F\dot{\psi}\,ds_x\,dt$$
$$\int_0^\infty e^{-2\sigma t}\int_\Gamma \left[\frac{p}{\alpha} + (V\frac{\partial p}{\partial t} - K\frac{\partial \varphi}{\partial t})\right] q\,ds_x\,dt = \int_0^\infty e^{-2\sigma t}\int_\Gamma \frac{Gq}{\alpha}\,ds_x\,dt\,.$$

The system can be written as

$$a(U,V) = l(V)\,, \tag{3.32}$$

where $U = (\varphi, p)$, $V = (\psi, q)$ and

$$a(U,V) = \int_0^\infty e^{-2\sigma t}\int_\Gamma \left(\alpha\dot{\varphi}\dot{\psi} + \frac{1}{\alpha}pq + K'p\dot{\psi} - W\varphi\dot{\psi} + V\dot{p}q - K\dot{\varphi}q\right) ds_x\,dt\,, \tag{3.33}$$

$$l(V) = \int_0^\infty e^{-2\sigma t}\int_\Gamma F\dot{\psi}\,ds_x\,dt + \int_0^\infty e^{-2\sigma t}\int_\Gamma \frac{Gq}{\alpha}\,ds_x\,dt\,. \tag{3.34}$$

Remark 3.4. *The system of equations* (3.31) *and the variational formulation* (3.32) *are the inverse Fourier-Laplace transforms of* (3.16) *and* (3.17).

Later we will also require the time–domain mapping properties of the boundary integral operators for general Sobolev exponents.

Theorem 3.6. *The following operators are continuous for* $s \in [-\frac{1}{2}, \frac{1}{2}]$, $r \in \mathbb{R}$*:*

$$\begin{aligned} V &: H^{r+1}_\sigma(\mathbb{R}^+, H^{s-\frac{1}{2}}(\Gamma)) \to H^r_\sigma(\mathbb{R}^+, H^{s+\frac{1}{2}}(\Gamma))\ , \\ K' &: H^{r+2}_\sigma(\mathbb{R}^+, H^{s-\frac{1}{2}}(\Gamma)) \to H^r_\sigma(\mathbb{R}^+, H^{s-\frac{1}{2}}(\Gamma))\ , \\ K &: H^{r+2}_\sigma(\mathbb{R}^+, H^{s+\frac{1}{2}}(\Gamma)) \to H^r_\sigma(\mathbb{R}^+, H^{s+\frac{1}{2}}(\Gamma))\ , \\ W &: H^{r+3}_\sigma(\mathbb{R}^+, H^{s+\frac{1}{2}}(\Gamma)) \to H^r_\sigma(\mathbb{R}^+, H^{s-\frac{1}{2}}(\Gamma))\ . \end{aligned}$$

Remark 3.5. *The* $+2$ *and* $+3$ *are still being improved.*

Proof. For $s = 0$ the theorem follows from Theorem 3.5 by applying the Fourier–Laplace transform. In fact, $r+2$ and $r+3$ may be replaced by $r+1$ in this case. The general case is discussed in a forthcoming preprint [20]. □

Together with Theorem 3.3, the mapping properties imply continuity and coercivity of the bilinear form $a(U,V)$.

Theorem 3.7. *Assume that* $\mathrm{Re}(\alpha_\infty) \geq 0$ *and* α, $\frac{1}{\alpha} \in L^\infty(\Gamma)$. *Then the bilinear form of the variational formulation* (3.32) *is continuous on* $\left(H^1_\sigma(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma)) \times H^1_\sigma(\mathbb{R}^+, L^2(\Gamma))\right) \times \left(H^1_\sigma(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma)) \times H^1_\sigma(\mathbb{R}^+, L^2(\Gamma))\right)$, *i.e., there exists* $C_\sigma > 0$ *such that:*

$$|a(U,V)| \leq C_\sigma(||p||_{1,0,\Gamma} + ||\varphi||_{1,\frac{1}{2},\Gamma})(||q||_{1,0,\Gamma} + ||\psi||_{1,\frac{1}{2},\Gamma})\,. \tag{3.35}$$

If $\mathrm{Re}(\alpha), \mathrm{Re}(\frac{1}{\alpha}) > 0$, *it verifies a coercivity estimate: There exists* $C_\sigma > 0$ *such that:*

$$a(U,U) \geq C_\sigma(||p||^2_{0,0,\Gamma} + ||\varphi||^2_{0,\frac{1}{2},\Gamma} + ||\dot{\varphi}||^2_{0,0,\Gamma})\,. \tag{3.36}$$

Proof. Equations (3.35) and (3.36) follow from Theorem 3.3 and Theorem 3.6. Concerning (3.36) we note that

$$\begin{aligned}
a(U,U) = |a(U,U)| &= |\int_{-\infty+i\sigma}^{\infty+i\sigma} a_\omega(\tilde{U},\tilde{U}) d\omega| \\
&\geq \int_{-\infty+i\sigma}^{\infty+i\sigma} \mathrm{Re}(a_\omega(\tilde{U},\tilde{U})) d\omega \\
&\geq C_\sigma \int_{-\infty+i\sigma}^{\infty+i\sigma} \left(||p(\omega)||^2_{0,\Gamma} + ||\varphi(\omega)||^2_{\frac{1}{2},\omega,\Gamma} + ||\omega\varphi(\omega)||^2_{0,\Gamma}\right) d\omega \\
&\geq C_\sigma(||p||^2_{0,0,\Gamma} + ||\varphi||^2_{0,\frac{1}{2},\Gamma} + ||\dot{\varphi}||^2_{0,0,\Gamma})\,.
\end{aligned}$$

Similary (3.35) is a consequence of (3.26) and Cauchy-Schwarz. □

Remark 3.6. *Similarly for the Dirichlet problem (see [22, (54)] and Corollary 3.50 in [21] for the full space)*

$$b(\varphi,\varphi) = \int_0^\infty \int_\Gamma V\dot{\varphi}(t,x)\varphi(t,x) d\sigma_x\, d_\sigma t \geq C_\sigma \|\varphi\|^2_{0,-\frac{1}{2},\Gamma} \tag{3.37}$$

4 Error Estimates

This chapter is divided into two parts. In the first part we derive a priori error estimates in space-time Sobolev spaces for the time domain boundary element method [23][21]. In the second part we consider the reliability of residual a posteriori error estimates. We use these a posteriori error estimates to define adaptive mesh refinements based on local error indicators. The residual error estimate is similar to the error estimate for elliptic boundary element problems, considered by Carstensen and Stephan [11] [12].

4.1 A priori Error Estimates

In this section we introduce two projection operators and their approximation properties, which are needed for the error analysis. Then we prove the convergence of Galerkin approximations for the Dirichlet and acoustic boundary problems in a half-space.
We begin by recalling the projection operators onto the space $H^k_\sigma(\Delta t, \mathbb{R})$ of piecewise polynomials of degree k in time [4].

Lemma 4.1. *(Lemma 3 in [4]) For $k < m$ let the operator $\Pi_{\Delta t} : \mathcal{H}^m_\sigma(\mathbb{R}^+, \mathbb{R}) \to H^k_\sigma(\Delta t, \mathbb{R})$ be defined by*

$$(\Pi_{\Delta t} f)(t) = f(t_n) + (t - t_n) f'(t_n) + \cdots + \frac{(t-t_n)^k}{k!} f^{(k)}(t_n) \quad \forall t_n \le t \le t_{n+1} \,.$$

Then we have

$$||f - \Pi_{\Delta t} f||_{\sigma,0} \le C_k \Delta t^{k+1} ||f||_{\sigma,k+1} \,.$$

Proof. We follow the proof in [4]. Taylor's formula with integral remainder asserts that

$$(f - \Pi_{\Delta t} f)(t) = \frac{1}{k!} \int_{t_n}^{t} (t-u)^k f^{(k+1)}(u) du \,, \qquad t \in I_n.$$

With the help of the Cauchy-Schwarz inequality we have

$$|f - \Pi_{\Delta t} f|^2 \le \frac{1}{(k!)^2 (2k+1)} (t - t_n)^{2k+1} \int_{t_n}^{t} (f^{(k+1)}(u))^2 du \,.$$

By multiplying the result by $e^{-2\sigma t}$ and integrating over the time intervall $[t_n, t_{n+1})$ we obtain

$$\int_{t_n}^{t_n+1} e^{-2\sigma t}|f(t)-(\Pi_{\Delta t}f)(t)|^2 dt \le \int_{t_n}^{t_n+1} \frac{1}{(k!)^2(2k+1)}(t-t_n)^{2k+1}\int_{t_n}^{t}(f^{(k+1)}(u))^2 du dt$$
$$\le \frac{1}{(k!)^2(2k+1)}(t-t_n)^{2k+1}\int_{t_n}^{t_n+1} e^{-2\sigma u}(f^{(k+1)}(u))^2 du$$
$$\cdot \int_{u}^{t_n+1} e^{-2\sigma(t-u)}(t-t_n)^{2k+1}dt\,.$$

Using

$$\int_{u}^{t_n+1} e^{-2\sigma(t-u)}(t-t_n)^{2k+1}dt \le \int_{u}^{t_n+1}(t-t_n)^{2k+1}dt \le \frac{\Delta t^{2k+2}}{2k+2}$$

we get

$$\int_{t_n}^{t_n+1} e^{-2\sigma t}|f(t)-(\Pi_{\Delta t}f)(t)|^2 dt \le \frac{\Delta t^{2k+2}}{(k!)^2(2k+1)(2k+2)}\int_{t_n}^{t_n+1} e^{-2\sigma u}(f^{(k+1)}(u))^2 du\,.$$

The summation over n gives us

$$\int_{0}^{\infty} e^{-2\sigma t}|f(t)-(\Pi_{\Delta t}f)(t)|^2 dt \le \sum_{n=0}^{\infty}\int_{t_n}^{t_n+1} e^{-2\sigma t}|f(t)-(\Pi_{\Delta t}f)(t)|^2 dt$$
$$\le \frac{\Delta t^{2k+2}}{(k!)^2(2k+1)(2k+2)}\sum_{n=0}^{\infty}\int_{t_n}^{t_n+1} e^{-2\sigma u}(f^{(k+1)}(u))^2 du$$
$$\le C_k\Delta t^{2k+2}\int_{0}^{\infty} e^{-2\sigma u}(f^{(k+1)}(u))^2 du\,.$$

□

We note the following consequences:

Corollary 4.1. *[4] Let $f \in \mathcal{H}_\sigma^{k+1}(\mathbb{R}^+,\mathbb{R})$. Then there holds for a constant C_k that*

$$||f-\Pi_{\Delta t}f||_{\sigma,\frac{1}{2}} \le C_k\Delta t^{k+\frac{1}{2}}||f||_{\sigma,k+1} \tag{4.1}$$

$$||f-\Pi_{\Delta t}f||_{\sigma,\frac{-1}{2}} \le C_k\Delta t^{k+\frac{3}{2}}||f||_{\sigma,k+1} \tag{4.2}$$

Proof. Because

$$(\Pi_{\Delta t}f)'(t) = f'(t_n)+(t-t_n)f''(t_n)+\cdots+\frac{(t-t_n)^{k-1}}{(k-1)!}f^{(k)}(t_n) = \Pi_{\Delta t}f'(t)$$

we obtain

$$||f-\Pi_{\Delta t}f||_{\sigma,1} = ||\partial_t(f-\Pi_{\Delta t})||_{\sigma,0}+||f-\Pi_{\Delta t}f||_{\sigma,0}$$
$$\le C_k\Delta t^{k}||f'||_{\sigma,k}+C_k\Delta t^{k+1}||f||_{\sigma,k}$$
$$\le C_k\Delta t^{k}||f||_{\sigma,k+1}+C_k\Delta t^{k+1}||f||_{\sigma,k+1}$$
$$\le C_k\Delta t^{k}||f||_{\sigma,k+1}\,.$$

By interpolation

$$\begin{aligned}||f - \Pi_{\Delta t} f||_{\sigma,\frac{1}{2}} &\leq ||f - \Pi_{\Delta t} f||^{\frac{1}{2}}_{\sigma,0} ||f - \Pi_{\Delta t} f||^{\frac{1}{2}}_{\sigma,1} \\ &\leq C_k \Delta t^k ||f||_{\sigma,k+1} .\end{aligned}$$

The proof of equation (4.2) is similar. □

While the above estimates have been discussed for the projection operator $\Pi_{\Delta t}$ onto the space of piecewise polynomials $H^k_\sigma(\Delta t, \mathbb{R})$, analogous estimates are easily derived for the interpolation operator $\widetilde{\Pi}_{\Delta t}$ onto the space of continuous, piecewise polynomials $V^k_{\Delta t}$.

For the discretization in space we recall:

Lemma 4.2. *(Lemma 4 in [4]) Let Π_h the orthogonal projection from $L^2(\Gamma)$ to V^p_h and $m \leq p$. Then*

$$\begin{aligned}||f - \Pi_h f||_{L^2(\Gamma)} &\leq C h^{m+1} ||f||_{H^{m+1}(\Gamma)} \\ ||f - \Pi_h f||_{H^{-\frac{1}{2}}(\Gamma)} &\leq C h^{m+\frac{3}{2}} ||f||_{H^{m+1}(\Gamma)}\end{aligned}$$

holds for all $f \in H^{m+1}(\Gamma)$.

Combining Π_h and $\Pi_{\Delta t}$ resp. $\widetilde{\Pi}_{\Delta t}$ one obtains with Proposition 3.54 in [21]:

Lemma 4.3. *Let $f \in H^s_\sigma(\mathbb{R}^+, H^m(\Gamma))$, $0 < m \leq q+1$, $0 < s \leq p+1$, $r \leq s$, $l \leq m$ such that $lr \geq 0$. Then if $l, r \leq 0$*

$$\|f - \Pi_h \circ \Pi_{\Delta t} f\|_{r,l,\Gamma} \leq C(h^\alpha + (\Delta t)^\beta) ||f||_{s,m,\Gamma} \ , \tag{4.3}$$

where $\alpha = \min\{m - l, m - \frac{m(l+r)}{m+s}\}$, $\beta = \min\{m + s - (l + r), m + s - \frac{m+s}{m} l\}$. If $l, r > 0$, $\beta = m + s - (l + r)$.

Likewise (4.3) holds with $\widetilde{\Pi}_{\Delta t}$ instead of $\Pi_{\Delta t}$, and on finite time intervals $[0, T]$ without the weight $e^{-\sigma t}$.

We are also going to require inverse estimates like (3.182) in [21] for $s, m \leq 0$

$$\|p_{h,\Delta t}\|_{0,0,\Gamma} \leq C(\Delta t)^s \max\left(h^m, \Delta t^m\right) \|p_{h,\Delta t}\|_{s,m,\Gamma}$$

for $p_{h,\Delta t}$ in the approximation spaces $V^{p,q}_{h,\Delta t}$, namely

$$\begin{aligned}||p_{h,\Delta t}||_{1,-\frac{1}{2},\Gamma} &\lesssim \frac{1}{\Delta t} ||p_{h,\Delta t}||_{0,-\frac{1}{2},\Gamma} &&\text{(in the proof of the Theorem 4.1)} \\ ||p_{h,\Delta t}||_{1,0,\Gamma} &\lesssim \frac{1}{\Delta t} ||p_{h,\Delta t}||_{0,0,\Gamma} &&\text{(in the proof of the Theorem 4.2)} \\ ||p_{h,\Delta t}||_{0,\frac{1}{2},\Gamma} &\lesssim \frac{1}{\sqrt{h}} ||p_{h,\Delta t}||_{0,0,\Gamma} &&\text{(in the proof of the Theorem 4.2).}\end{aligned}$$

The above inverse inequalities hold due to the standard estimates for regular finite element functions in the usual Sobolev-spaces $H^s(\Gamma)$ [3] on one hand, and on the other hand the weight function $e^{-\sigma t}$ does not affect these inequalies (see [4, Lemma 2]), c.f. (3.177) in [21].

4.1.1 Dirichlet Boundary Value Problem in a Half-Space

We now use the above approximation results to discuss the convergence of Galerkin approximations to the Dirichlet problem. Consider the variational form of the boundary integral equation $V\phi = f$:
Find $\phi \in H^1_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma))$ such that

$$b(\phi, \psi) = \langle \dot{f}, \psi \rangle \qquad \forall \psi \in H^1_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma))\,, \tag{4.4}$$

where

$$\begin{aligned} b(\phi, \psi) &= \int_0^\infty \int_\Gamma V\dot{\phi}(t,x)\psi(t,x)ds_x\, d_\sigma t \\ \langle \dot{f}, \psi \rangle &= \int_0^\infty \int_\Gamma \dot{f}(t,x)\psi(t,x)ds_x\, d_\sigma t\,. \end{aligned}$$

The Galerkin equations read:
Find $\phi_{h,\Delta t} \in V^{p,q}_{h,\Delta t}$ such that

$$b(\phi_{h,\Delta t}, \psi_{h,\Delta t}) = l(\psi_{h,\Delta t}) = \int_0^\infty e^{-2\sigma t} \int_\Gamma \dot{f}_{h,\Delta t}(t,x)\psi_{h,\Delta t}(t,x)ds_x\, dt \qquad \forall \psi_{h,\Delta t} \in V^{p,q}_{h,\Delta t} \tag{4.5}$$

Theorem 4.1. *For the solution* $\phi \in H^1_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}(\Gamma)})$ *of* (4.4) *and* $\phi_{h,\Delta t} \in V^{p,q}_{h,\Delta t}$ *of* (4.5) *there holds:*

$$\begin{aligned} \|\phi - \phi_{h,\Delta t}\|_{0,-\frac{1}{2},\Gamma} \leq C \Bigg(& ||f_{h,\Delta t} - f||_{1,\frac{1}{2},\Gamma} \\ & + \inf_{\psi_{h,\Delta t}} \{(1 + \frac{1}{\Delta t})\|\phi - \psi_{h,\Delta t}||_{0,-\frac{1}{2},\Gamma} + \frac{1}{\Delta t}\|\dot{\phi} - \dot{\psi}_{h,\Delta t}||_{0,-\frac{1}{2},\Gamma}\} \Bigg)\,. \end{aligned}$$

If in addition $H^s_\sigma(\mathbb{R}^+, H^m(\Gamma))$, *then*

$$\begin{aligned} \|\phi - \phi_{h,\Delta t}\|_{0,-\frac{1}{2},\Gamma} \leq C \Bigg(& ||f_{h,\Delta t} - f||_{1,\frac{1}{2},\Gamma} \\ & + \left((h^{\alpha_1} + \Delta t^{\beta_1})(1 + \frac{1}{\Delta t}) + (h^{\alpha_2} + \Delta t^{\beta_2})\frac{1}{\Delta t} \right) ||\phi||_{s,m,\Gamma} \Bigg) \end{aligned}$$

where

$$\begin{aligned} \alpha_1 &= \min\{m + \frac{1}{2}, m - \frac{m}{2(m+s)}\}, \beta_1 = \min\{m + s + \frac{1}{2}, m + s + \frac{m+s}{2m}\}\,, \\ \alpha_2 &= \min\{m + \frac{1}{2}, m - \frac{m}{2(m+s-1)}\}, \beta_2 = \min\{m + s - \frac{1}{2}, m + s - 1 + \frac{m+s-1}{2m}\}\,, \end{aligned}$$

and $\quad m \geq -\frac{1}{2},\ s \geq 0.$

Proof. We start with the coercivity applied to $\phi_{h,\Delta t} - \psi_{h,\Delta t} \in H^1_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma))$ with $\psi_{h,\Delta t} \in V_{h,\Delta t}$:

$$C\|\phi_{h,\Delta t} - \psi_{h,\Delta t}\|^2_{0,-\frac{1}{2},\Gamma} \leq b(\phi_{h,\Delta t} - \phi, \phi_{h,\Delta t} - \psi_{h,\Delta t}) + b(\phi - \psi_{h,\Delta t}, \phi_{h,\Delta t} - \psi_{h,\Delta t})\,.$$

For the first term we obtain:

$$\begin{aligned} b(\phi_{h,\Delta t}-\phi, \phi_{h,\Delta t}-\psi_{h,\Delta t}) &= \int_0^\infty e^{-2\sigma t}\int_\Gamma (\dot f_{h,\Delta t}-\dot f)(\phi_{h,\Delta t}-\psi_{h,\Delta t})\, ds_x\, dt \\ &\le \|\dot f_{h,\Delta t}-\dot f\|_{0,\frac12,\Gamma}\|\phi_{h,\Delta t}-\psi_{h,\Delta t}\|_{0,-\frac12,\Gamma}\,. \end{aligned}$$

By continuity (Corollary 3.51 in [21]) we can estimate the second term as follows:

$$b(\phi-\psi_{h,\Delta t}, \phi_{h,\Delta t}-\psi_{h,\Delta t}) \le C\|\phi-\psi_{h,\Delta t}\|_{1,-\frac12,\Gamma}\|\phi_{h,\Delta t}-\psi_{h,\Delta t}\|_{1,-\frac12,\Gamma}\,.$$

The inverse inequality in the time variable leads to

$$b(\phi-\psi_{h,\Delta t}, \phi_{h,\Delta t}-\psi_{h,\Delta t}) \le \frac{C}{\Delta t}\|\phi-\psi_{h,\Delta t}\|_{1,-\frac12,\Gamma}\|\phi_{h,\Delta t}-\psi_{h,\Delta t}\|_{0,-\frac12,\Gamma}\,,$$

so that finally we get:

$$\|\phi_{h,\Delta t}-\psi_{h,\Delta t}\|_{0,-\frac12,\Gamma} \le C\{||f_{h,\Delta t}-f||_{1,\frac12,\Gamma}+\frac{1}{\Delta t}\|\phi-\psi_{h,\Delta t}||_{1,-\frac12,\Gamma}\}\,.$$

With the triangle inequality one shows that

$$\begin{aligned} \|\phi-\phi_{h,\Delta t}\|_{0,-\frac12,\Gamma} &\le \|\phi-\psi_{h,\Delta t}\|_{0,-\frac12,\Gamma}+\|\phi_{h,\Delta t}-\psi_{h,\Delta t}\|_{0,-\frac12,\Gamma} \\ &\le C\{||f_{h,\Delta t}-f||_{1,\frac12,\Gamma}+\inf_{\psi_{h,\Delta t}}\{\|\phi-\psi_{h,\Delta t}||_{0,-\frac12,\Gamma}+\frac{1}{\Delta t}\|\phi-\psi_{h,\Delta t}||_{1,-\frac12,\Gamma}\}\} \\ &\le C\Big(||f_{h,\Delta t}-f||_{1,\frac12,\Gamma} \\ &\qquad +\inf_{\psi_{h,\Delta t}}\{(1+\frac{1}{\Delta t})\|\phi-\psi_{h,\Delta t}||_{0,-\frac12,\Gamma}+\frac{1}{\Delta t}\|\dot\phi-\dot\psi_{h,\Delta t}||_{0,-\frac12,\Gamma}\}\Big) \end{aligned}$$

The second inequality follows from the approximation properties stated in Lemma 4.3.

□

4.1.2 Acoustic Boundary Value Problem

Next, we consider the Galerkin equations for (3.32), i.e. find $\Phi_{h,\Delta t}=(p_{h,\Delta t},\varphi_{h,\Delta t})\in V^{\tilde p,\tilde q}_{h,\Delta t}\times V^{p,q}_{h,\Delta t}$ such that $\forall\Psi_{h,\Delta t}=(q_{h,\Delta t},\psi_{h,\Delta t})\in V^{\tilde p,\tilde q}_{h,\Delta t}\times V^{p,q}_{h,\Delta t}$

$$a(\Phi_{h,\Delta t},\Psi_{h,\Delta t})=\tilde l(\Psi_{h,\Delta t}):=\int_0^\infty e^{-2\sigma t}\int_\Gamma F_{h,\Delta t}\dot\psi_{h,\Delta t}\,ds_x\,dt+\int_0^\infty e^{-2\sigma t}\int_\Gamma \frac{G_{h,\Delta t}q_{h,\Delta t}}{\alpha}\,ds_x\,dt\,. \tag{4.6}$$

We now derive an estimate for the error of the above Galerkin approximation to (3.32) in the norm $|||.|||$ defined by:

$$|||\Phi|||=\Big(||p||^2_{0,0,\Gamma}+||\varphi||^2_{0,\frac12,\Gamma}+||\dot\varphi||^2_{0,0,\Gamma}\Big)^{\frac12}\quad\forall\Phi=(p,\varphi)\,.$$

Theorem 4.2. *Assume that* $Re\,\alpha_\infty, Re\,\alpha \geq 0$ *and* $\alpha, \frac{1}{\alpha} \in L^\infty(\Gamma)$. *For the solutions* $\Phi = (p, \varphi) \in H^1_\sigma(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma)) \times H^1_\sigma(\mathbb{R}^+, L^2(\Gamma))$ *of* (3.32) *and* $\Phi_{h,\Delta t} = (p_{h,\Delta t}, \varphi_{h,\Delta t})$ *of* (4.6) *there holds:*

$$
\begin{aligned}
&|||(p - p_{h,\Delta t}, \varphi - \varphi_{h,\Delta t})||| \\
&\leq C_\sigma \left(||F_{h,\Delta t} - F||_{0,0,\Gamma} + ||G_{h,\Delta t} - G||_{0,0,\Gamma}\right) \\
&+ C_\sigma \max\left(\frac{1}{\Delta t}, \frac{1}{\sqrt{h}}\right) \inf_{(q_{h,\Delta t},\psi_{h,\Delta t}) \in V^{\tilde{p},\tilde{q}}_{h,\Delta t} \times V^{p,q}_{h,\Delta t}} \left(||p - q_{h,\Delta t}||_{1,0,\Gamma} + ||\varphi - \psi_{h,\Delta t}||_{1,\frac{1}{2},\Gamma}\right).
\end{aligned}
$$

If in addition $\varphi \in H^{s_1}_\sigma(\mathbb{R}^+, H^{m_1}(\Gamma))$, $p \in H^{s_2}_\sigma(\mathbb{R}^+, H^{m_2}(\Gamma))$, *then we have*

$$
\begin{aligned}
|||(p - p_{h,\Delta t}, \varphi - \varphi_{h,\Delta t})||| &\leq C_\sigma \left(||F_{h,\Delta t} - F||_{0,0,\Gamma} + ||G_{h,\Delta t} - G||_{0,0,\Gamma}\right) \\
&+ C_\sigma \max\left(\frac{1}{\Delta t}, \frac{1}{\sqrt{h}}\right) \left((h^{\alpha_1} + \Delta t^{\beta_1})||p||_{s_1,m_1,\Gamma} + (h^{\alpha_2} + \Delta t^{\beta_2})||\varphi||_{s_2,m_2,\Gamma}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_1 &= m_1 \quad , \quad \beta_1 = m_1 + s_1 - 1\,, \\
\alpha_2 &= \min\{m_2 - \frac{1}{2}, m_2 - \frac{3m_2}{2(m_2 + s_2)}\}, \quad \beta_2 = m_2 + s_2 - \frac{3}{2}\,.
\end{aligned}
$$

Proof. We write $\Phi = (p, \varphi)$ and $\Psi = (q, \psi)$. We start with the coercivity (3.36) applied to $\Phi_{h,\Delta t} - \Psi_{h,\Delta t} \in H^1_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma))$ and $\psi_{h,\Delta t}$:

$$
\begin{aligned}
C(\Gamma)|||\Phi_{h,\Delta t} - \Psi_{h,\Delta t}|||^2 &= C(\Gamma)\Big(\|p_{h,\Delta t} - q_{h,\Delta t}\|^2_{0,0,\Gamma} + \|\varphi_{h,\Delta t} - \psi_{h,\Delta t}\|^2_{\sigma,0,\frac{1}{2},\Gamma} \\
&\qquad + \|\dot{\varphi}_{h,\Delta t} - \dot{\psi}_{h,\Delta t}\|^2_{0,0,\Gamma}\Big) \\
&\leq a(\Phi_{h,\Delta t} - \Psi_{h,\Delta t}, \Phi_{h,\Delta t} - \Psi_{h,\Delta t}) \\
&= a(\Phi_{h,\Delta t} - \Phi, \Phi_{h,\Delta t} - \Psi_{h,\Delta t}) + a(\Phi - \Psi_{h,\Delta t}, \Phi_{h,\Delta t} - \Psi_{h,\Delta t})
\end{aligned}
$$

Taking into account (3.32) and (3.33)-(3.34), we obtain for the first term :

$$
\begin{aligned}
a(\Phi_{h,\Delta t} - \Phi, \Phi_{h,\Delta t} - \Psi_{h,\Delta t}) &= \int_0^\infty e^{-2\sigma t} \int_\Gamma (F_{h,\Delta t} - F)(\dot{\varphi}_{h,\Delta t} - \dot{\psi}_{h,\Delta t})\, ds_x\, dt \\
&+ \int_0^\infty e^{-2\sigma t} \int_\Gamma (\frac{G_{h,\Delta t}}{\alpha} - \frac{G}{\alpha})(p_{h,\Delta t} - q_{h,\Delta t})\, ds_x\, dt \\
&\lesssim ||F_{h,\Delta t} - F||_{0,0,\Gamma}||\dot{\varphi}_{h,\Delta t} - \dot{\psi}_{h,\Delta t}||_{0,0,\Gamma} \\
&\qquad + ||G_{h,\Delta t} - G||_{0,0,\Gamma}||p_{h,\Delta t} - q_{h,\Delta t}||_{0,0,\Gamma} \\
&\leq \left(||F_{h,\Delta t} - F||_{0,0,\Gamma} + ||G_{h,\Delta t} - G||_{0,0,\Gamma}\right) \\
&\qquad \left(||\dot{\varphi}_{h,\Delta t} - \dot{\psi}_{h,\Delta t}||_{0,0,\Gamma} + ||p_{h,\Delta t} - q_{h,\Delta t}||_{0,0,\Gamma}\right) \\
&\leq \left(||F_{h,\Delta t} - F||_{0,0,\Gamma} + ||G_{h,\Delta t} - G||_{0,0,\Gamma}\right)|||\Phi_{h,\Delta t} - \Psi_{h,\Delta t}|||\,.
\end{aligned}
$$

Due to the continuity (3.35) we can estimate the second term by

$$
\begin{aligned}
|a(\Phi - \Psi_{h,\Delta t}, \Phi_{h,\Delta t} - \Psi_{h,\Delta t})| &\leq C_\sigma \left(||p - q_{h,\Delta t}||_{1,0,\Gamma} + ||\varphi - \psi_{h,\Delta t}||_{1,\frac{1}{2},\Gamma}\right) \\
&\quad \cdot \left(||\varphi_{h,\Delta t} - \psi_{h,\Delta t}||_{1,\frac{1}{2},\Gamma} + ||p_{h,\Delta t} - q_{h,\Delta t}||_{1,0,\Gamma}\right).
\end{aligned}
\tag{4.7}
$$

Taking into account the inverse estimate from the first section of this chapter, we have

$$\begin{aligned} ||\varphi_{h,\Delta t} - \psi_{h,\Delta t}||_{1,\frac{1}{2}} &\lesssim ||\varphi_{h,\Delta t} - \psi_{h,\Delta t}||_{0,\frac{1}{2},\Gamma} + ||\dot{\varphi}_{h,\Delta t} - \dot{\psi}_{h,\Delta t}||_{0,\frac{1}{2},\Gamma} \\ &\lesssim ||\varphi_{h,\Delta t} - \psi_{h,\Delta t}||_{0,\frac{1}{2},\Gamma} + (h^{-\frac{1}{2}} + \Delta t^{-\frac{1}{2}})||\dot{\varphi}_{h,\Delta t} - \dot{\psi}_{h,\Delta t}||_{0,0,\Gamma} \end{aligned} \tag{4.8}$$

and in time

$$||p_{h,\Delta t} - q_{h,\Delta t}||_{1,0,\Gamma} \leq \frac{1}{\Delta t}||p_{h,\Delta t} - q_{h,\Delta t}||_{0,0,\Gamma}\,. \tag{4.9}$$

Now putting (4.8) and (4.9) in (4.7) we get

$$\begin{aligned} |a(\Phi - \Psi_{h,\Delta t}, \Phi_{h,\Delta t} - \Psi_{h,\Delta t})| &\leq C_\sigma \left(||p - q_{h,\Delta t}||_{1,0,\Gamma} + ||\varphi - \psi_{h,\Delta t}||_{1,\frac{1}{2},\Gamma}\right) \\ &\cdot \Bigg(||\varphi_{h,\Delta t} - \psi_{h,\Delta t}||_{0,\frac{1}{2},\Gamma} + (\frac{1}{\sqrt{h}} + \frac{1}{\sqrt{\Delta t}})||\dot{\varphi}_{h,\Delta t} - \dot{\psi}_{h,\Delta t}||_{0,0,\Gamma} \\ &\qquad + \frac{1}{\Delta t}||p_{h,\Delta t} - q_{h,\Delta t}||_{0,0,\Gamma}\Bigg) \\ &\leq C_\sigma \max\left(\frac{1}{\Delta t}, \frac{1}{\sqrt{h}}\right)\left(||p - q_{h,\Delta t}||_{1,0,\Gamma} + ||\varphi - \psi_{h,\Delta t}||_{1,\frac{1}{2},\Gamma}\right) \\ &\qquad \cdot|||\Phi_{h,\Delta t} - \Psi_{h,\Delta t}|||\,. \end{aligned}$$

Altogether, we conclude

$$\begin{aligned} |||\Phi - \Phi_{h,\Delta t}||| &\leq C_\sigma \left(||F_{h,\Delta t} - F||_{0,0,\Gamma} + ||G_{h,\Delta t} - G||_{0,0,\Gamma}\right) \\ &+ C_\sigma \max\left(\frac{1}{\Delta t}, \frac{1}{\sqrt{h}}\right)\left(||p - q_{h,\Delta t}||_{1,0,\Gamma} + ||\varphi - \psi_{h,\Delta t}||_{1,\frac{1}{2},\Gamma}\right) \\ &+ |||\Phi - \Psi_{h,\Delta t}|||\,. \end{aligned} \tag{4.10}$$

Using the interpolation operator from Lemma 4.4, we obtain the powers of h and Δt stated in the theorem.

□

4.2 A Simple Residual A posteriori Estimate for TDBIE

In this section, we study a general framework for a posteriori error estimates in the time domain boundary element method. An adaptive procedure is desired where the algorithm itself decides when and where to refine the mesh in order to improve the computed Galerkin solution.
One such strategy has been proposed by Carstensen and Stephan in the 1990s, whose residual a posteriori error estimate we generalize to the hyperbolic case.
Consider the variational form of the wave equation in $\mathbb{R}^3_+ \setminus \Omega$ on the finite time–interval $[0,T]$:
Find $\phi \in H^1_0([0,T], H^{-\frac{1}{2}}(\Gamma)$ such that for all $\psi \in H^1_0([0,T], H^{-\frac{1}{2}}(\Gamma))$

$$b_T(\phi, \psi) = \langle \dot{f}, \psi \rangle\;, \tag{4.11}$$

with

$$b_T(\phi,\psi) = \int_0^T \int_\Gamma V\dot{\phi}(t,x)\ \psi(t,x)\ ds_x\, dt\ ,$$
$$\langle \dot{f},\psi\rangle = \int_0^T \int_\Gamma \dot{f}(t,x)\ \psi(t,x)\ ds_x\, dt\ .$$

As shown in [22], the bilinear form b_T is continuous, and also weakly coercive after averaging in T:

Proposition 4.1. *For every $\phi,\psi \in H^1_0([0,T],H^{-\frac{1}{2}}(\Gamma))$ there holds:*

$$|b_T(\phi,\psi)| \leq C\|\phi\|_{1,-\frac{1}{2},\Gamma}\|\psi\|_{1,-\frac{1}{2},\Gamma}$$

and

$$\|\phi\|^2_{0,-\frac{1}{2},\Gamma} \leq C\int_0^T b_t(\phi,\phi)\ dt = C\int_0^T \int_0^t \int_\Gamma V\dot{\phi}(s,x)\ \psi(s,x)\ ds_x\, ds\ dt\ .$$

Proof. Continuity of the bilinear form is a consequence of the mapping properties in Theorem 3.6, adapted to the finite interval $[0,T]$.
The coercivity estimate can be found in Equation (59) of [22], using slightly different notation. □

Solving the continuous problem (4.11) and its Galerkin discretization

$$b_T(\phi_{h,\Delta t},\psi_{h,\Delta t}) = \langle \dot{f},\psi_{h,\Delta t}\rangle\ , \tag{4.12}$$

in $V^{p,q}_{h,\Delta t}$, we will make use of Galerkin orthogonality:

$$b_T(\phi-\phi_{h,\Delta t},\psi_{h,\Delta t}) = 0 \qquad \forall \psi_{h,\Delta t}\in V^{p,q}_{h,\Delta t}\,.$$

Using ideas for the elliptic problem we conclude the following a posteriori estimate:

Theorem 4.3. *Let $\phi,\phi_{h,\Delta t} \in H^1_0([0,T],H^{-\frac{1}{2}}(\Gamma))$ be the solutions to (4.11) and its discretized variant, and assume that $R=\dot{f}-V\dot{\phi}_{h,\Delta t}\in H^0([0,T],H^1(\Gamma))$. Then*

$$\begin{aligned}\|\phi-\phi_{h,\Delta t}\|^2_{0,-\frac{1}{2},\Gamma} &\lesssim \|R\|_{0,1,\Gamma}\big(\Delta t\|\partial_t R\|_{0,0,\Gamma}+\|h\cdot\nabla R\|_{0,0,\Gamma}\big)\\ &\lesssim \max\{\Delta t,h\}\big(\|\partial_t R\|_{L^2([0,T],L^2(\Gamma))}+\|\nabla R\|_{L^2([0,T],L^2(\Gamma))}\big)^2\end{aligned}$$

Remark 4.1. *a) As the single–layer potential maps $H^1([0,T],L^2(\Gamma))$ continuously to $H^0([0,T],H^1(\Gamma))$, $V\dot{\phi}_{h,\Delta t}$ belongs to $H^0([0,T],H^1(\Gamma))$ if, for example, $\phi_{h,\Delta t}\in H^2([0,T],L^2(\Gamma))$. The a posteriori estimate is therefore only valid for sufficiently smooth discretizations, e.g. constructed from C^1–continuous splines.*
b) In practice, we will use $\Delta t\|\partial_t R\|_{0,0,\Gamma}+\|h\cdot\nabla R\|_{0,0,\Gamma}$ as an error indicator.

Proof of Theorem 4.3. We first note that for all $\psi_{h,\Delta t} \in V^{p,q}_{h,\Delta t}$

$$
\begin{aligned}
\|\phi - \phi_{h,\Delta t}\|^2_{0,-\frac{1}{2},\Gamma} &\lesssim \int_0^T b_t(\phi - \phi_{h,\Delta t}, \phi - \phi_{h,\Delta t})\ dt \\
&= \int_0^T \int_0^t \int_\Gamma \dot{f}(\phi - \phi_{h,\Delta t})\ ds_x\ ds\ dt - \int_0^T b_t(\phi_{h,\Delta t}, \phi - \phi_{h,\Delta t})\ dt \\
&= \int_0^T \int_0^t \int_\Gamma \dot{f}(\phi - \psi_{h,\Delta t})\ ds_x\ ds\ dt - \int_0^T b_t(\phi_{h,\Delta t}, \phi - \psi_{h,\Delta t})\ dt \\
&= \int_0^T \int_0^t \int_\Gamma (\dot{f} - V\dot{\phi}_{h,\Delta t})(\phi - \psi_{h,\Delta t})\ ds_x\ ds\ dt\ .
\end{aligned}
$$

The last term may be estimated by:

$$
\begin{aligned}
&\int_0^T \int_0^t \int_\Gamma (\dot{f} - V\dot{\phi}_{h,\Delta t})(\phi - \psi_{h,\Delta t})\ ds_x\ ds\ dt \\
&= \int_0^T (T-s) \int_\Gamma (\dot{f} - V\dot{\phi}_{h,\Delta t})(\phi - \psi_{h,\Delta t})\ ds_x\ ds \\
&\le T\ \|R\|_{0,\frac{1}{2},\Gamma} \|\phi - \psi_{h,\Delta t}\|_{0,-\frac{1}{2},\Gamma}\ .
\end{aligned}
$$

We use $\psi_{h,\Delta t} = \phi_{h,\Delta t}$ together with the interpolation inequality

$$
\|R\|^2_{0,\frac{1}{2},\Gamma} \le \|R\|_{0,0,\Gamma} \|R\|_{0,1,\Gamma}\ .
$$

As the residual is perpendicular to $V^{p,q}_{h,\Delta t}$,

$$
\begin{aligned}
\|R\|^2_{0,0,\Gamma} &= \langle R, R\rangle = \langle R, R - \widetilde{\psi}_{h,\Delta t}\rangle \\
&\le \|R\|_{0,0,\Gamma} \|R - \widetilde{\psi}_{h,\Delta t}\|_{0,0,\Gamma}
\end{aligned}
$$

for all $\widetilde{\psi}_{h,\Delta t} \in V^{p,q}_{h,\Delta t}$, we obtain

$$
\|R\|_{0,0,\Gamma} \le \inf\{\|R - \widetilde{\psi}_{h,\Delta t}\|_{0,0,\Gamma} : \widetilde{\psi}_{h,\Delta t} \in V^{p,q}_{h,\Delta t}\}\ .
$$

Choosing $\widetilde{\psi}_{h,\Delta t} = \widetilde{\Pi}_{h,\Delta t} R$, based on the interpolation operator defined at the beginning of this chapter, we obtain

$$
\|R\|_{0,0,\Gamma} \lesssim \Delta t \|\partial_t R\|_{0,0,\Gamma} + \|h \cdot \nabla R\|_{0,0,\Gamma}\ .
$$

The theorem follows. □

4.3 An A Posteriori Error Estimate for the Acoustic Problem

The analysis of the acoustic boundary problem can be done in a similar way.
For a finite time interval $[0, T]$ we introduce the bilinear form

$$
a_T((\varphi, p), (\psi, q)) = \int_0^T \int_\Gamma \left(\alpha \dot{\varphi} \dot{\psi} + \frac{1}{\alpha} pq + K' p \dot{\psi} - W \varphi \dot{\psi} + V \dot{p} q - K \dot{\varphi} q \right) ds_x\, dt\,. \tag{4.13}
$$

analogous to (3.33) for $T = \infty$. With

$$l(\psi, q) = \int_0^T \int_\Gamma F\dot{\psi}\, ds_x\, dt + \int_0^\infty \int_\Gamma \frac{Gq}{\alpha}\, ds_x\, dt\,, \tag{4.14}$$

we consider the variational formulation for the wave equation in $\mathbb{R}^3$ with acoustic boundary conditions on Γ:
Find $(\varphi, p) \in H^1([0,T], H^{\frac{1}{2}}(\Gamma)) \times H^1([0,T], L^2(\Gamma))$ such that

$$a_T((\varphi, p), (\psi, q)) = l(\psi, q) \tag{4.15}$$

for all $(\psi, q) \in H^1([0,T], H^{\frac{1}{2}}(\Gamma)) \times H^1([0,T], L^2(\Gamma))$.

Again we observe from (3.36) that the bilinear form a_T satisfies a weak coercivity estimate, even without averaging in T (cf. (66) of [22]):

$$\|p\|^2_{0,0,\Gamma} + \|\dot{\varphi}\|^2_{0,0,\Gamma} + \|\varphi\|^2_{0,\frac{1}{2},\Gamma} \lesssim a_T((\varphi, p), (\varphi, p))\ .$$

For now, we only observe a simple a posteriori estimate.

Theorem 4.4. *Let* $(\varphi, p), (\varphi_{h,\Delta t}, p_{h,\Delta t}) \in H^1_0([0,T], H^{\frac{1}{2}}(\Gamma)) \times H^1([0,T], L^2(\Gamma))$ *be the solutions to* (4.15) *and its discretized variant, and assume that*

$$\begin{aligned} R_1 &= F - \alpha\dot{\varphi}_{h,\Delta t} - K' p_{h,\Delta t} + W\varphi_{h,\Delta t} \in L^2([0,T], L^2(\Gamma))\ , \\ R_2 &= \frac{G}{\alpha} - \alpha^{-1} p_{h,\Delta t} - V\dot{p}_{h,\Delta t} + K\dot{\varphi}_{h,\Delta t} \in L^2([0,T], L^2(\Gamma))\ . \end{aligned}$$

Then

$$\begin{aligned} &\|p - p_{h,\Delta t}\|_{0,0,\Gamma} + \|\dot{\varphi} - \dot{\varphi}_{h,\Delta t}\|_{0,0,\Gamma} + \|\varphi - \varphi_{h,\Delta t}\|_{0,\frac{1}{2},\Gamma} \\ &\qquad \lesssim \|R_1\|_{0,0,\Gamma} + \|R_2\|_{0,0,\Gamma}\ . \end{aligned}$$

Proof. For every $(\psi_{h,\Delta t}, q_{h,\Delta t}) \in V_h^p \otimes V_{\Delta t}^q$ as in Theorem 3.7 we have

$$\begin{aligned} &\|p - p_{h,\Delta t}\|^2_{0,0,\Gamma} + \|\dot{\varphi} - \dot{\varphi}_{h,\Delta t}\|^2_{0,0,\Gamma} + \|\varphi - \varphi_{h,\Delta t}\|^2_{0,\frac{1}{2},\Gamma} \\ &\lesssim a_T((\varphi - \varphi_{h,\Delta t}, p - p_{h,\Delta t}), (\varphi - \varphi_{h,\Delta t}, p - p_{h,\Delta t})) \\ &= \langle R_1, \dot{\varphi} - \dot{\psi}_{h,\Delta t}\rangle + \langle R_2, p - q_{h,\Delta t}\rangle \\ &\leq \|R_1\|_{0,0,\Gamma} \|\dot{\varphi} - \dot{\psi}_{h,\Delta t}\|_{0,0,\Gamma} \\ &\qquad + \|R_2\|_{0,0,\Gamma} \|p - q_{h,\Delta t}\|_{0,0,\Gamma} \\ &\leq (\|R_1\|_{0,0,\Gamma} + \|R_2\|_{0,0,\Gamma}) \times \\ &\qquad (\|p - q_{h,\Delta t}\|_{0,0,\Gamma} + \|\dot{\varphi} - \dot{\psi}_{h,\Delta t}\|_{0,0,\Gamma})\ . \end{aligned}$$

The assertion is obtained by choosing $(\psi_{h,\Delta t}, q_{h,\Delta t}) = (\varphi_{h,\Delta t}, p_{h,\Delta t})$. □

Remark 4.2. *Naturally, under stronger assumptions on R_1, R_2 we may obtain powers of h and Δt on the right hand side by the following argument:*
As in the proof of Theorem 4.3, $\langle R_1, \dot{\widetilde{\psi}}_{h,\Delta t}\rangle = \langle R_2, \widetilde{q}_{h,\Delta t}\rangle = 0$ for all $(\widetilde{\psi}_{h,\Delta t}, \widetilde{q}_{h,\Delta t}) \in V_h^p \otimes V_{\Delta t}^q$. Hence

$$\begin{aligned}\|R_2\|^2_{0,0,\Gamma} &= \langle R_2, R_2\rangle = \langle R_2, R_2 - \widetilde{q}_{h,\Delta t}\rangle \\ &\leq \|R_2\|_{0,0,\Gamma}\|R_2 - \widetilde{q}_{h,\Delta t}\|_{0,0,\Gamma}.\end{aligned}$$

Choosing $\widetilde{q}_{h,\Delta t} = \Pi_{h,\Delta t} R_2$ yields e.g.

$$\|R_2\|_{0,0,\Gamma} \lesssim \Delta t\|\partial_t R_2\|_{0,0,\Gamma} + \|h\cdot\nabla_\Gamma R_2\|_{0,0,\Gamma} + \Delta t\|h\cdot\nabla_\Gamma\partial_t R_2\|_{0,0,\Gamma}$$

provided $R_2 \in H^1([0,T], H^1(\Gamma))$.
Assuming $R_1 \in H^1([0,T], H^1(\Gamma))$, we similarly have

$$\begin{aligned}\|R_1\|^2_{0,0,\Gamma} &= \langle R_1, R_1\rangle = \langle R_1, R_1 - \dot{\widetilde{\psi}}_{h,\Delta t}\rangle \\ &\leq \|R_1\|_{\frac{1}{2}-s,0,\Gamma}\|\int_0^t R_1 - \widetilde{\psi}_{h,\Delta t}\|_{\frac{1}{2}+s,0,\Gamma} \ .\end{aligned}$$

Choosing $\widetilde{\psi}_{h,\Delta t} = \Pi_{h,\Delta t}\int_0^t R_1$ and $s = \frac{1}{2}$ results in

$$\begin{aligned}\|R_1\|_{0,0,\Gamma} \lesssim\ &\Delta t\|\partial_t R_1\|_{0,0,\Gamma} + \|h\cdot\nabla_\Gamma R_1\|_{0,0,\Gamma} \\ &+ \Delta t\|h\cdot\nabla_\Gamma\partial_t R_1\|_{0,0,\Gamma} \ .\end{aligned}$$

Altogether,

$$\begin{aligned}&\|p - p_{h,\Delta t}\|_{0,0,\Gamma} + \|\dot\varphi - \dot\varphi_{h,\Delta t}\|_{0,0,\Gamma} + \|\varphi - \varphi_{h,\Delta t}\|_{0,\frac{1}{2},\Gamma} \\ &\lesssim \|R_1\|_{0,0,\Gamma} + \|R_2\|_{0,0,\Gamma} \\ &\lesssim \sum_{i=1}^{2} \Delta t\|\partial_t R_i\|_{0,0,\Gamma} + \|h\cdot\nabla R_i\|_{0,0,\Gamma} \\ &\qquad + \Delta t\|h\cdot\nabla\partial_t R_i\|_{0,0,\Gamma} \ .\end{aligned}$$

4.4 Numerical Example

We consider the Dirichlet problem for the wave equation in the exterior of the three-dimensional unit ball with a singular right hand side. In the formulation

$$V\dot\varphi(t,x) := \frac{1}{4\pi}\int_\Gamma \frac{\dot\varphi(t-|x-y|, y)}{|x-y|}ds_y = \dot f(t,x)$$

as an integral equation of the first kind on the sphere Γ, we choose the right-hand side

$$\dot f(t,x) = \begin{cases} 2, & x_1 > 0 \\ 0, & x_1 < 0 \end{cases} .$$

The function $\dot{f}$ is a toy example for a time–independent singularity, similar to the singular horn-like geometry where a tyre meets a street. We expect adaptive mesh refinements to concentrate around the line of discontinuity of $\dot{f}$, given by $x_1 = 0$.
Starting with an initial coarse mesh T_0, we use the MOT scheme from Section 2.2 to obtain a Galerkin approximation to this equation. This approximate solution is given by:

$$\dot{\varphi}_{h,\Delta t}(x,t) = \sum_{m=1}^{N_t} \sum_{i=1}^{N_s} b_i^m \beta^m(t) \varphi_i(x)$$

where β^m is the linear hat function

$$\beta^m(t) = (\Delta t)^{-1}((t - t_m)\chi_m - (t - t_{m+1})\chi_{m+1})$$

and φ_i is the piecewise linear hat function associated to node i.

From $\dot{\varphi}_{h,\Delta t}$ and $\dot{f}$ we determine in every triangle $\triangle$ and every time interval $I_n = [t_{n-1}, t_n]$ the local error indicator

$$\eta_\triangle(I_n)^2 = \int_{t_{n-1}}^{t_n} \int_\triangle [h \nabla_\Gamma (\dot{f} - V\dot{\varphi}_{h,\Delta t})]^2 \,,$$

where the time integral is approximated by a Riemann sum.
As a first step towards a space–time adaptive Galerkin method, we concentrate on space-adaptive mesh refinements based on the time-integrated indicator

$$\eta_\triangle = \sqrt{\sum_{n=1}^{N_t} \eta_\triangle(I_n)^2} \;.$$

The term in the above a posteriori estimates, which involves the time–derivative of the residual $\dot{f} - V\dot{\varphi}_{h,\Delta t}$, is neglected in this example, because we expect $\eta_\triangle(I_n)$ to dominate for this time–independent singularity.
To compute $\eta_\triangle$ from $\dot{\varphi}_{h,\Delta t}$, we consider the gradient of $V\dot{\varphi}_{h,\Delta t}$ as a singular integral:

$$\nabla V\dot{\varphi}(t,x) = \frac{1}{4\pi} \int_\Gamma (x-y) \left(\frac{\dot{\varphi}_{h,\Delta t}(t - |x-y|, y)}{|x-y|^3} + \frac{\ddot{\varphi}_{h,\Delta t}(t - |x-y|, y)}{|x-y|^2} \right) ds_y \,.$$

Explicitly,

$$\begin{aligned}\nabla V\dot{\varphi}_{h,\Delta t}(t,x) = \frac{1}{4\pi} \sum_{m=1}^{N_t} \sum_{i=1}^{N_s} b_i^m \Bigg[\int_\Gamma \varphi_i(y) \Bigg(& \beta^m(t - |x-y|) \frac{x-y}{|x-y|^3} \\ & + \dot{\beta}^m(t - |x-y|) \frac{x-y}{|x-y|^2} \Bigg) ds_y \Bigg]\end{aligned}$$

with

$$\begin{aligned}
\int_\Gamma \varphi_i(y)\dot\beta^m(t-|x-y|)\frac{x-y}{|x-y|^2}\,ds_y
&= \frac{1}{\Delta t}\int_\Gamma \varphi_i(y)\left(H(t-t_{m-1}-|x-y|)-H(t-t_m-|x-y|)\right)\frac{x-y}{|x-y|^2}\,ds_y\\
&\quad -\frac{1}{\Delta t}\int_\Gamma \varphi_i(y)\left(H(t-t_m-|x-y|)-H(t-t_{m+1}-|x-y|)\right)\frac{x-y}{|x-y|^2}\,ds_y\\
&= \frac{1}{\Delta t}\int_{t-t_m\le|x-y|\le t-t_{m-1}} \varphi_i(y)\frac{x-y}{|x-y|^2}\,ds_y\\
&\qquad -\frac{1}{\Delta t}\int_{t-t_{m+1}\le|x-y|\le t-t_m} \varphi_i(y)\frac{x-y}{|x-y|^2}\,ds_y
\end{aligned}$$

and

$$\begin{aligned}
\int_\Gamma \varphi_i(y)\beta^m(t-|x-y|)\frac{x-y}{|x-y|^3}\,ds_y &= \frac{t-t_{m-1}}{\Delta t}\int_{t-t_m\le|x-y|\le t-t_{m-1}} \varphi_i(y)\frac{x-y}{|x-y|^3}\,ds_y\\
&\quad -\frac{t-t_{m+1}}{\Delta t}\int_{t-t_{m+1}\le|x-y|\le t-t_m} \varphi_i(y)\frac{x-y}{|x-y|^3}\,ds_y\,.
\end{aligned}$$

Here we calculate the integral over the complicated intersection of $\triangle$ with the light–cone based on the routines also used to set–up the Galerkin matrix for V. From $\nabla V\dot\varphi_{h,\Delta t}(t,x)$ we obtain the surface gradient

$$\nabla_\Gamma(\dot f(t,x)-V\dot\varphi_{h,\Delta t}(t,x)) = \nabla(\dot f(t,x)-V\dot\varphi_{h,\Delta t}(t,x)) - \vec n.(\vec n.\nabla(\dot f(t,x)-V\dot\varphi_{h,\Delta t}(t,x))).$$

The error indicators $\eta_\triangle$ lead to an adaptive algorithm, based on the 4 steps

$$\textbf{SOLVE} \longrightarrow \textbf{ESTIMATE} \longrightarrow \textbf{MARK} \longrightarrow \textbf{REFINE}.$$

Adaptive Algorithm:
Input: Spatial mesh $T = T_0$, refinement parameter $\theta \in (0,1)$, tolerance $\epsilon > 0$, data f.

1. Solve $V\dot\varphi_{h,\Delta t} = \dot f$ on T.
2. Compute the error indicators $\eta(\triangle)$ in each triangle $\triangle \in T$.
3. Find $\eta_{\max} = \max_\triangle \eta(\triangle)$.
4. Stop if $\sum_i \eta^2(\triangle_i) < \epsilon^2$.
5. Mark all $\triangle \in T$ with $\eta(\triangle_i) > \theta\eta_{\max}$.
6. Refine each marked triangle into 4 new triangles to obtain a new mesh T (and project the new nodes onto the sphere). Choose Δt such that $\frac{\Delta t}{\Delta x} \le 1$ for all traingles.
7. Go to 1.

Output: Approximation of φ.
According to the a posteriori estimates derived in this Chapter, the error between the approximate and the actual solution to the problem is bounded by a multiple of ϵ, up to quantities involving time–derivatives of the residual $\dot{f} - V\dot{\varphi}_{h,\Delta t}$.
The numerical experiment depicted in Figure 4.1 shows the first three meshes generated by the above adaptive algorithm, starting with an initial icosahedral triangulation of the sphere with 80 nodes. Most refinements are near the discontinuity of f, as expected.

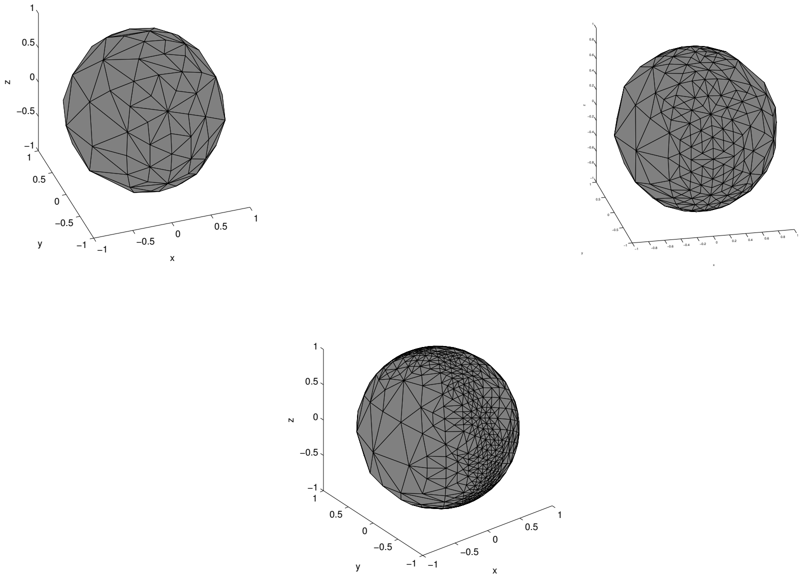


Figure 4.1: The first three adaptively generated meshes for $V\dot{\varphi} = \dot{f}$ starting from an icosahedral triangulation with 80 nodes, $\theta = 0.9$.

Unfortunately, the practical evaluation of $\nabla_\Gamma V\dot{\varphi}_{h,\Delta t}(t,x))$ as a singular integral is not efficient. The computational time to calculate the error indicators in each triangle and each time steps takes approximately 6 CPU seconds on standard desktop computer. While this is easily parallelized, future work might instead investigate a finite difference approximation of $\nabla_\Gamma V\dot{\varphi}_{h,\Delta t}$.
The above experiment presents only a first step towards space–time adaptive TDBEM, for the case of the geometric singularities relevant to sound radiation of tyres. Fully space–time adaptive methods have been explored by M. Gläfke [21]. The optimal use of space–time adaptivity and its application to the acoustic boundary conditions in Section 4.3 remain to be explored.

5 Discretization and Numerical Experiments

5.1 Numerical Discretization

5.1.1 The Dirichlet Problem in the Absorbing Half-Space

In this section we examine the exterior Dirichlet problem for the wave equation in the half-space by using boundary integral methods. Consider the exterior Dirichlet problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \quad & \text{in } \mathbb{R}^+ \times \Omega^e \\ u(0,x) = \frac{\partial u}{\partial t}(0,x) = 0 \quad & \\ u = f \quad & \text{on } \mathbb{R}^+ \times \Gamma \\ \frac{\partial u}{\partial n} - \alpha_\infty \frac{\partial u}{\partial t} = 0 \quad & \text{on } \mathbb{R}^+ \times \Gamma_\infty = \{z = 0\}\,. \end{aligned} \tag{5.1}$$

As noted before, the Dirichlet problem is equivalent to finding a solution to the boundary integral equation

$$V\varphi(t,x) = f(t,x) \qquad \text{for all} \quad (t,x) \in \mathbb{R}^+ \times \Gamma\,. \tag{5.2}$$

We have already seen in Chapter 3 that the single layer potential V in the half-space is given by:

$$\begin{aligned} V\varphi(t,x) &= V_1\varphi(t,x) + V_2\varphi(t,x) + V_3\varphi(t,x) \\ &= \frac{1}{2\pi}\int_\Gamma \frac{p(t-|x-y|,y)}{|x-y|} ds_y + \frac{1}{2\pi}\int_\Gamma \frac{p(t-|x-y'|,y)}{|x-y'|} ds_y \\ &\quad - \frac{\alpha_\infty}{\pi}\int_0^\infty \int_\Gamma \frac{\partial}{\partial s}\Big[\frac{H(t-s-|x-y'|)}{\sqrt{(t-s+\alpha_\infty\vartheta_3)^2 + (\alpha_\infty^2 - 1)R^2}}\Big] p(s,y)\, ds_y\, ds\,, \end{aligned}$$

where $\vartheta_3 = x_3 + y_3$ and $R^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2$.
V_1 is the single layer potential in the whole space $\mathbb{R}^3$ and was previously considered in thesis of Ostermann [40]. The term V_2 is V_1 composed with a reflection at Γ_∞, so that $V_1 + V_2$ corresponds to the wave equation with homogeneous Neumann conditions on Γ_∞. More general boundary conditions involve V_3. We define the bilinear form

associated with the single layer potential:

$$a(\varphi, q) = \int_0^\infty e^{-2\sigma t} \int_\Gamma V_1\varphi(t,x)\dot{q}(t,x) ds_x dt + \int_0^\infty e^{-2\sigma t} \int_\Gamma V_2\varphi(t,x)\dot{q}(t,x) ds_x dt$$
$$+ \int_0^\infty e^{-2\sigma t} \int_\Gamma V_3\varphi(t,x)\dot{q}(t,x) ds_x dt \,.$$

The weak formulation of the Dirichlet problem then is

$$a(\varphi, q) = \int_0^\infty e^{-2\sigma t} \int_\Gamma f(t,x)\dot{q}(t,x) ds_x dt \qquad \text{for all test functions } q \,.$$

For the numerical approximation we consider a Galerkin discretization in space and time. As ansatz functions we use tensor products of piecewise constant functions in time and piecewise constant functions in space.
For the second term we write $\varphi_{h,\Delta t}(t,x) = \sum_{m=1}^{N_t} \sum_{i=1}^{N_s} b_i^m \varphi_i(x)\gamma_m(t)$ where γ_m and φ_i are the basis in time and in space. Then for $q(t,x) = \gamma_n(t)\varphi_j(x)$ we obtain

$$\int_0^\infty \int_\Gamma V_2\varphi_{h,\Delta t}(t,x)\dot{q}(t,x) ds_x dt = \sum_{m=1}^{N_t} \sum_{i=1}^{N_s} b_i^m \int_\Gamma \int_\Gamma \Upsilon_0^{n-m}(x,y) \frac{\varphi_i(y)\varphi_j(x)}{|x-y'|} ds_y ds_x \,,$$

where $\Upsilon_0^{n-m}(x,y)$ is defined by

$$\begin{aligned}\Upsilon_0^{n-m}(x,y) &= \int_0^\infty \gamma_m(t-|x-y'|)\dot{\gamma}_n(t)\, dt \\ &= \chi_{E'_{n-m-1}}(x,y) - \chi_{E'_{n-m}}(x,y) \,,\end{aligned}$$

and the indicator function $\chi_A(x)$ for a set A is given by

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} .$$

The reflected acoustic cone E'_l is given by

$$E'_l := \left\{ (x,y) \in \Gamma \times \Gamma \text{ s.t. } t_l \leq |x-y'| \leq t_{l+1} \right\} .$$

Consequently, we have

$$\int_0^\infty \int_\Gamma V_2\varphi_{h,\Delta t}(t,x)\dot{q}(t,x) ds_x dt = \sum_{m=1}^{N_t} \sum_{i=1}^{N_s} b_i^m \left(\int\int_{E'_{n-m-1}} \frac{\varphi_i(y)\varphi_j(x)}{|x-y'|} ds_y ds_x - \int\int_{E'_{n-m}} \frac{\varphi_i(y)\varphi_j(x)}{|x-y'|} ds_y ds_x \right) . \tag{5.3}$$

To calculate $\int\int_{E_l'} \frac{\varphi_i(y)\varphi_j(x)}{|x-y'|} ds_y ds_x$ we basically follow the lines of [40]:
Let x in $\mathbb{R}^3_+$,

$$E'(x) := B_{r_{\max}}(x') \setminus B_{r_{\min}}(x') = \left\{ y \in \mathbb{R}^3 \text{ s.t. } r_{\min} \leq |x' - y| \leq r_{\max} \right\} ,$$

the light cone or domain of influence corresponding to $x' = (x_1, x_2, -x_3)$, $r_{\min} := t_l$ and $r_{\max} := t_{l+1}$. Then

$$E_l' := \bigcup_{x \in \Gamma} E'(x) \, .$$

We rewrite each of the integrals in (5.3) as

$$G_{ij}^{\nu} = \sum_{\substack{T_{i'} \subset \operatorname{supp} \varphi_i \\ T_{j'} \subset \operatorname{supp} \varphi_j}} \int_{T_{j'}} \varphi_j(x) P_{i,i'}(x) \, ds_x \, ,$$

with a retarded potential $P_{i,i'}$ given by

$$P_{i,i'}(x) := \int_{E'(x) \cap T_{i'}} \frac{1}{|x-y|} \varphi_i(y) \, ds_y \, . \tag{5.4}$$

Decomposition of the domain of integration $E'(x) \cap T$

We seek a parametric representation of the domain of integration $E'(x) \cap T$. The domain of influence $E'(x)$ of the point x' in T is an annular domain with center x' and radii $r_{\min}$ and $r_{\max}$. Therefore, we have to integrate over those points in the triangle T which lie between two concentric spheres. The three-dimensional intersection can be rewritten as a two-dimensional intersection in the plane of the triangle within a three-dimensional space. Let x_t' denote the orthogonal projection of x' onto the plane $\mathcal{E}_T$ containing T and define $d := |x' - x_t'|$, cf. Fig. 5.2. Then

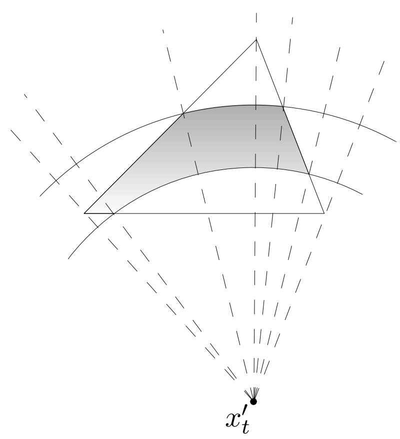


Figure 5.1: Example for a decomposition of $E'(x) \cap T$ with respect to x_t' into $n_d = 5$ subelements

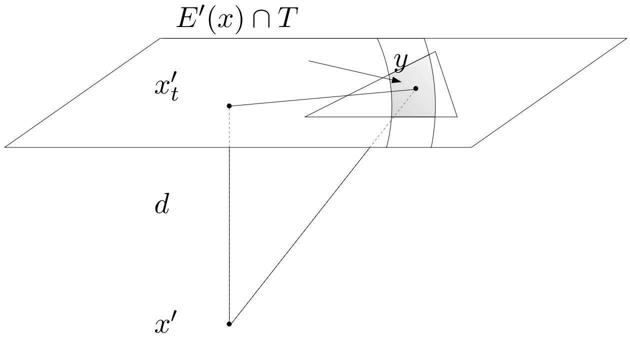


Figure 5.2: Projection of x' onto the plane $\mathcal{E}_T$

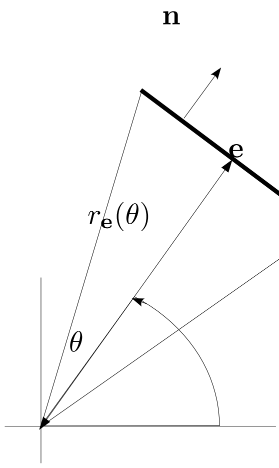


Figure 5.3: Representation of an edge e in polar coordinates (r, θ).

$$E'(x) \cap \mathcal{E}_T = (B_{r'_{\min}}(x'_t) \setminus B_{r'_{\max}}(x'_t)) \cap \mathcal{E}_T = \left\{ y \in \mathcal{E}_T \ : \ r'_{\min} \leq |x'_t - y| \leq r'_{\max} \right\},$$

where $r'_{\min/\max} := (r^2_{\min/\max} - d^2)^{1/2}$. Thus

$$E'(x) \cap T = (B_{r'_{\min}}(x'_t) \setminus B_{r'_{\max}}(x'_t)) \cap T\,.$$

Now we introduce polar coordinates (r, θ) around x'_t and decompose $E'(x) \cap T = \bigcup_{l=1}^{n_d} D_l$. The subdomains D_l have the form (see Figure 5.4):

$$D_l := \{(r, \theta) \ : \ \theta \in (\theta_l, \theta_{l+1}) \text{ and } r \in (r_{1,l}(\theta), r_{2,l}(\theta))\}\ .$$

The radial limits $r_{1,l}$ and $r_{2,l}$ are given by

$$r_{1,l} := \begin{cases} r'_{\min} & e \in B_{r'_{\min}(x)} \\ r_e(\theta) & \text{else} \end{cases} \qquad\qquad r_{2,l} := \begin{cases} r'_{\max} & e \notin B_{r'_{\max}(x)} \\ r_e(\theta) & \text{else} \end{cases}\ ,$$

where $r_e(\theta)$ is

$$r_e(\theta) = \frac{v \cdot n}{n_1 \cos\theta + n_2 \sin\theta}\,, \tag{5.5}$$

as in Figure 5.3. Here $n = (n_1, n_2, n_3)^T$ denotes the normal to the edge and v is any point on e.
The integral (5.4) then is the sum of integrals over the geometrically simpler pieces D_l.

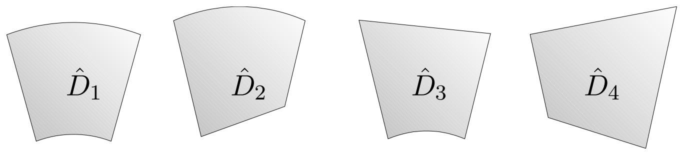


Figure 5.4: Generic integration domains

We now consider the entries of the Galerkin matrix of the third term

$$\int_0^\infty \int_\Gamma V_3 \varphi_{h,\Delta t}(t,x)\dot{q}(t,x) ds_x dt =$$
$$\int_0^\infty \int_\Gamma \int_0^\infty \int_\Gamma -\frac{\alpha_\infty}{\pi}\frac{\partial}{\partial s}\left[\frac{H(t-s-|x-y'|)}{\sqrt{(t-s+\alpha_\infty \vartheta_3)^2 + (\alpha_\infty^2 - 1)R^2}}\right] \varphi_{h,\Delta t}(s,y)\dot{q}(t,x) ds_y\, ds_x ds\, dt\,.$$

Using piecewise constant basis functions in time and space as above we obtain the matrix elements:

$$V_{3,ijmn} = -\frac{\alpha_\infty}{\pi}\int_0^\infty \int_\Gamma \int_0^\infty \int_\Gamma \frac{\partial}{\partial s}\left[\frac{H(t-s-|x-y'|)}{\sqrt{(t-s+\alpha_\infty \vartheta_3)^2 + (\alpha_\infty^2 - 1)R^2}}\right]$$
$$\times\, \gamma_m(s)\varphi_i(y)\dot{\gamma}_n(t)\varphi_j(x) ds_y\, ds_x ds\, dt\,.$$

As before we evaluate the time integration analytically. Let us first discuss the evaluation of time integrals with one retarded time argument, i.e. given two functions f_1 and f_2, we need to calculate integrals of the form

$$\int_0^\infty \int_0^\infty f_1(t-s-|x-y|) f_2(t)\, dt\, ds. \tag{5.6}$$

They occur in the computation of the Galerkin entries of the discrete space-time variational formulations discussed earlier.
Before we examine the retarded time integral more generally, we discuss a simple model integral to clarify the chosen approach. Choose a piecewise constant basis function in time as a sum of Heavyside functions

$$\gamma^m(t) = \chi_{\mathcal{I}_m}(t) = H(t - t_{m-1}) - H(t - t_m)\,.$$

Using integration by parts, we obtain with $\gamma^m(0) = 0$:

$$
\begin{aligned}
&- \frac{\alpha_\infty}{\pi} \int_0^\infty \int_0^\infty \frac{\partial}{\partial s} \left[\frac{H(t-s-|x-y'|)}{\sqrt{(t-s+\alpha_\infty \vartheta_3)^2 + (\alpha_\infty^2 - 1)R^2}} \right] \gamma^m(s) \dot{\gamma}^n(t)\, dt\, ds \\
&= \frac{\alpha_\infty}{\pi} \int_0^\infty \int_0^\infty \left[\frac{H(t-s-|x-y'|)}{\sqrt{(t-s+\alpha_\infty \vartheta_3)^2 + (\alpha_\infty^2 - 1)R^2}} \right] \dot{\gamma}^m(s) \dot{\gamma}^n(t)\, dt\, ds \\
&= \frac{\alpha_\infty}{\pi} \int_0^\infty \left[\int_0^\infty \frac{H(t-s-|x-y'|))(\delta(s-t_{m-1}) - \delta(s-t_m))}{\sqrt{(t-s+\alpha_\infty \vartheta_3)^2 + (\alpha_\infty^2 - 1)R^2}}\, ds \right] \times \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\delta(t-t_{n-1}) - \delta(t-t_n))\, dt \\
&= \frac{\alpha_\infty}{\pi} \int_0^\infty \left[\frac{H(t-t_{m-1}-|x-y'|)}{\sqrt{(t-t_{m-1}+\alpha_\infty \vartheta_3)^2 + (\alpha_\infty^2 - 1)R^2}} \right] (\delta(t-t_{n-1}) - \delta(t-t_n))\, dt \\
&\quad - \frac{\alpha_\infty}{\pi} \int_0^\infty \left[\frac{H(t-t_m-|x-y'|)}{\sqrt{(t-t_m+\alpha_\infty \vartheta_3)^2 + (\alpha_\infty^2 - 1)R^2}} \right] (\delta(t-t_{n-1}) - \delta(t-t_n))\, dt \\
&= \frac{\alpha_\infty}{\pi} \Bigg(\frac{H(t_{n-1}-t_{m-1}-|x-y'|)}{\sqrt{(t_{n-1}-t_{m-1}+\alpha_\infty \vartheta_3)^2 + (\alpha_\infty^2 - 1)R^2}} - \frac{H(t_n-t_{m-1}-|x-y'|)}{\sqrt{(t_n-t_{m-1}+\alpha_\infty \vartheta_3)^2 + (\alpha_\infty^2 - 1)R^2}} \\
&\quad - \frac{H(t_{n-1}-t_m-|x-y'|)}{\sqrt{(t_{n-1}-t_m+\alpha_\infty \vartheta_3)^2 + (\alpha_\infty^2 - 1)R^2}} + \frac{H(t_n-t_m-|x-y'|)}{\sqrt{(t_n-t_m+\alpha_\infty \vartheta_3)^2 + (\alpha_\infty^2 - 1)R^2}} \Bigg) \\
&= \frac{\alpha_\infty}{\pi} \Bigg(- \frac{H(t_{n-m-1}-|x-y'|)}{\sqrt{(t_{n-m-1}+\alpha_\infty \vartheta_3)^2 + (\alpha_\infty^2 - 1)R^2}} + 2 \frac{H(t_{n-m}-|x-y'|)}{\sqrt{(t_{n-m}+\alpha_\infty \vartheta_3)^2 + (\alpha_\infty^2 - 1)R^2}} \\
&\quad - \frac{H(t_{n-m+1}-|x-y'|)}{\sqrt{(t_{n-m+1}+\alpha_\infty \vartheta_3)^2 + (\alpha_\infty^2 - 1)R^2}} \Bigg) .
\end{aligned}
$$

We define the four-dimensional set

$$
K'_l := \left\{ (x,y) \in \Gamma \times \Gamma \,:\, |x-y'| \le t_l \right\} ,
$$

and write

$$
L(l,x,y) = \frac{\alpha_\infty}{\pi} \frac{1}{\sqrt{(t_l+\alpha_\infty \vartheta_3)^2 + (\alpha_\infty^2 - 1)R^2}} .
$$

Because $K'_{n-m-1} \subset K'_{n-m} \subset K'_{n-m+1}$, the time-integral becomes

$$
\begin{aligned}
-L(n-m-1,x,y)\chi_{K'_{n-m-1}}(x,y) + 2L(n-m,x,y)\chi_{K'_{n-m}}(x,y) \\
- L(n-m+1,x,y)\chi_{K'_{n-m+1}}(x,y) .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
V_{3,ijmn} = &- \int\int_{K'_{n-m-1}} L(n-m-1,x,y)\varphi_i(y)\varphi_j(x)ds_y\, ds_x \\
&+ 2\int\int_{K'_{n-m}} L(n-m,x,y)\varphi_i(y)\varphi_j(x)ds_y\, ds_x \\
&- \int\int_{K'_{n-m+1}} L(n-m+1,x,y)\varphi_i(y)\varphi_j(x)ds_y\, ds_x\,.
\end{aligned}
$$

The integrals over K'_l are computed like V_1 and V_2 with $r_{min} = 0$ and with kernels $L(l,x,y)$.

5.1.2 The Neumann Problem in the Absorbing Half-Space

Similarly to the Dirichlet, in this section we consider the Neumann problem for the wave equation in the half-space:

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - \Delta u &= 0 \quad \text{in } \mathbb{R}^+ \times \Omega^e \\
u(0,x) &= \frac{\partial u}{\partial t}(0,x) = 0 \\
\frac{\partial u}{\partial n} &= f \quad \text{on } \mathbb{R}^+ \times \Gamma \\
\frac{\partial u}{\partial n} - \alpha_\infty \frac{\partial u}{\partial t} &= 0 \quad \text{on } \mathbb{R}^+ \times \Gamma_\infty\,.
\end{aligned}
\tag{5.7}
$$

The corresponding integral equation of the second kind is

$$
(-I + K')\varphi(t,x) = 2f(t,x)\,, \tag{5.8}
$$

where

$$
\begin{aligned}
K'\varphi(t,x) = &\frac{1}{2\pi}\int_\Gamma \frac{\partial}{\partial n_x}\left(\frac{\varphi(t-|x-y|,y)}{|x-y|}ds_y\right) + \frac{1}{2\pi}\int_\Gamma \frac{\partial}{\partial n_x}\left(\frac{\varphi(t-|x-y'|,y)}{|x-y'|}\right)ds_y \\
&- \frac{\alpha_\infty}{\pi}\int_0^\infty\int_\Gamma \frac{\partial}{\partial n_x}\left(\frac{\partial}{\partial \tau}[\frac{H(t-\tau-|x-y'|)}{\sqrt{(t-\tau+\alpha_\infty\vartheta_3)^2 + (\alpha_\infty^2-1)R^2}}]\varphi(\tau,y)\right)ds_y d\tau\,.
\end{aligned}
$$

The numerical implementation involves the term

$$
\Sigma(t-\tau,x,y) = -\frac{\alpha_\infty}{\pi}\frac{\partial}{\partial t}[\frac{H(t-\tau-|x-y'|)}{\sqrt{(t-\tau+\alpha_\infty\vartheta_3)^2 + (\alpha_\infty^2-1)R^2}}]\,. \tag{5.9}
$$

We define

$$
A(\tau) := \sqrt{(t-\tau+\alpha_\infty\vartheta_3)^2 + (\alpha_\infty^2-1)R^2}
$$

and obtain from (5.9)

$$
\begin{aligned}
\frac{\partial \Sigma}{\partial x_1} &= -\frac{\alpha_\infty}{\pi}\frac{\partial}{\partial t}\left[\frac{\frac{\partial}{\partial y_1}(H(t-\tau-|x-y'|))}{A(\tau)} - \frac{\frac{\partial A(\tau)}{\partial y_1}H(t-\tau-|x-y'|)}{A(\tau)^2}\right] \\
&= \frac{\alpha_\infty}{\pi}\frac{\partial}{\partial t}\left[(x_1-y_1)\left(\frac{\delta(t-\tau-|x-y'|)}{|x-y'|A(\tau)} + \frac{\alpha_\infty^2-1}{A(\tau)^3}H(t-\tau-|x-y'|)\right)\right]\,,
\end{aligned}
$$

$$\frac{\partial \Sigma}{\partial x_2} = \frac{\alpha_\infty}{\pi} \frac{\partial}{\partial t} \left[(x_2 - y_2) \left(\frac{\delta(t - \tau - |x - y'|)}{|x - y'| A(\tau)} + \frac{\alpha_\infty^2 - 1}{A(\tau)^3} H(t - \tau - |x - y'|) \right) \right] ,$$

$$\frac{\partial \Sigma}{\partial x_3} = \frac{\alpha_\infty}{\pi} \frac{\partial}{\partial t} \left[(x_3 + y_3) \left(\frac{\delta(t - \tau - |x - y'|)}{|x - y'| A(\tau)} + \frac{t - \tau + \alpha_\infty \vartheta_3}{A(\tau)^3} H(t - \tau - |x - y'|) \right) \right] .$$

Therefore,

$$\begin{aligned} \frac{\partial \Sigma}{\partial n_x} &= \frac{\alpha_\infty}{\pi} \frac{\partial}{\partial t} [\frac{(x - y') \cdot n_x}{|x - y'| A(\tau)} \delta(t - \tau - |x - y'|) \\ &\quad + \frac{(\alpha_\infty^2 - 1) \left((x_1 - y_1) n_1 + (x_2 - y_2) n_2 \right) + (t - \tau + \alpha_\infty \vartheta_3) n_3}{A(\tau)^3} H(t - \tau - |x - y'|)] . \end{aligned}$$

We define

$$B(\tau) = \frac{(\alpha_\infty^2 - 1) \left((x_1 - y_1) n_1 + (x_2 - y_2) n_2 \right) + (t - \tau + \alpha_\infty \vartheta_3) n_3}{A(\tau)^3} .$$

As for V we write the adjoint double layer potential as a sum of three terms:

$$K' \varphi(t, x) = K_1' \varphi(t, x) + K_2' \varphi(t, x) + K_3' \varphi(t, x)$$

where

$$\begin{aligned} K_3' \varphi(t, x) &= \int_0^\infty \int_\Gamma \frac{\partial \Sigma}{\partial n_x} \varphi(\tau, y) d\tau dy \\ &= \frac{\alpha_\infty}{\pi} \int_0^\infty \int_\Gamma \frac{\partial}{\partial \tau} [\frac{(x - y') . n_x}{|x - y'| A(\tau)} \delta(t - \tau - |x - y'|) \\ &\qquad + B(\tau) H(t - \tau - |x - y'|)] \varphi(\tau, y) d\tau dy . \end{aligned}$$

Convolution and integration by parts lead to

$$\begin{aligned} K_3' \varphi(t, x) = -\frac{\alpha_\infty}{\pi} \int_\Gamma \frac{(x - y') . n_x}{|x - y'| A(\tau)} \frac{\partial \varphi}{\partial t} (t - |x - y'|, y) dy \\ - \frac{\alpha_\infty}{\pi} \int_0^\infty \int_\Gamma B(\tau) H(t - \tau - |x - y'|)] \frac{\partial \varphi(\tau, y)}{\partial \tau} d\tau dy . \end{aligned}$$

We note that the first term in K_3' has the same form as V_2 with a different kernel function,

$$\frac{(x - y') . n_x}{|x - y'| A(\tau)} ,$$

and its discretization is similar to the one of V_2.
For piecewise constant ansatz and test functions in time we get

$$\int\limits_{t_{n-1}}^{t_n}\int\limits_{t_{m-1}}^{t_m}[B(\tau)H(t-\tau-|x-y'|)](\delta(\tau-t_{m-1})-\delta(\tau-t_m))d\tau dt$$
$$=\int\limits_{t_{n-1}}^{t_n}[B(t_{m-1})H(t-t_{m-1}-|x-y'|)]dt$$
$$-\int\limits_{t_{n-1}}^{t_n}[B(t_m)H(t-t_m-|x-y'|)]dt\,.$$

We call the first integral $Z_1(m,n)$ and the second $Z_2(m,n) = Z_1(m+1,n)$. $K'_{3,ijmn}$ becomes:

$$\begin{aligned}K'_{3,ijmn} = -\frac{\alpha_\infty}{\pi}\Bigg(\int\int_{E'_{n-m-1}}&\frac{(x-y').n_x}{|x-y'|A(\tau)}\varphi_i(y)\varphi_j(x)ds_y ds_x\\ &-\int\int_{E'_{n-m}}\frac{(x-y').n_x}{|x-y'|A(\tau)}\varphi_i(y)\varphi_j(x)ds_y ds_x\Bigg)\\ &-\frac{\alpha_\infty}{\pi}\int_\Gamma\left[Z_1(m,n)-Z_2(m,n)\right]\varphi_i(y)\varphi_j(x)ds_y ds_x\,.\end{aligned}$$

Now, we want to analyze $Z_1(m,n)$ and $Z_2(m,n)$. With the substitution
$u_j = t-t_j+\alpha_\infty\vartheta_3$ and $a^2 = (\alpha_\infty^2-1)R^2$ we obtain

$$\begin{aligned}&\int\limits_{t_{n-1}}^{t_n}[\frac{(\alpha_\infty^2-1)\left((x_1-y_1)n_1+(x_2-y_2)n_2\right)+(t-t_j+\alpha_\infty\vartheta_3)n_3}{A(t_j)^3}H(t-t_j-|x-y'|)]dt\\ &=(\alpha_\infty^2-1)\left((x_1-y_1)n_1+(x_2-y_2)n_2\right)\int_{\tau_j^1}^{\tau_j^2}\frac{1}{(u_j^2+a^2)^{\frac{3}{2}}}du_j+\int_{\tau_j^3}^{\tau_j^4}\frac{u_j}{(u_j^2+a^2)^{\frac{3}{2}}}du_j\\ &=(\alpha_\infty^2-1)\left((x_1-y_1)n_1+(x_2-y_2)n_2\right)\left[\frac{u_j}{a^2\sqrt{u_j^2+a^2}}\right]_{\tau_j^1}^{\tau_j^2}+n_3\left[\frac{1}{a^2\sqrt{u_j^2+a^2}}\right]_{\tau_j^3}^{\tau_j^4}\end{aligned}$$

For the evaluation of the $Z_1(m,n)$ and $Z_2(m,n)$ we distinguish 6 different cases, depending on $r = |x-y'|$.

case 1 : $t_n < t_{m-1}+r$

$$Z_1(m,n)=0 \qquad \text{and} \qquad Z_2(m,n)=0$$

case 2 : $t_{n-1} < t_{m-1}+r < t_n < t_m+r$

$$Z_1(m,n)=(\alpha_\infty^2-1)\left((x_1-y_1)n_1+(x_2-y_2)n_2\right)\left[\frac{u}{a^2\sqrt{u^2+a^2}}\right]_{t_{m-1}+r}^{t_n}+n_3\left[\frac{1}{a^2\sqrt{u^2+a^2}}\right]_{t_{m-1+r}}^{t_n}$$

and $Z_2(m,n) = 0$, $t_{n-1} - t_{m-1} < r < t_n < t_n - t_{m-1}$
case 3 : $t_{m-1} + r < t_{n-1} < t_n < t_m + r$

$$Z_1(m,n) = (\alpha_\infty^2 - 1)\left((x_1 - y_1)n_1 + (x_2 - y_2)n_2\right)\left[\frac{u}{a^2\sqrt{u^2+a^2}}\right]_{t_{n-1}}^{t_n} + n_3\left[\frac{1}{a^2\sqrt{u^2+a^2}}\right]_{t_{n-1}}^{t_n}$$

and $Z_2(m,n) = 0$, $t_n - t_{m+1} < r < t_n < t_{n-1} - t_{m-1}$
case 4 : $t_{m-1} + r < t_{n-1} < t_m + r < t_n$

$$Z_1(m,n) = (\alpha_\infty^2 - 1)\left((x_1 - y_1)n_1 + (x_2 - y_2)n_2\right)\left[\frac{u}{a^2\sqrt{u^2+a^2}}\right]_{t_{n-1}}^{t_n} + n_3\left[\frac{1}{a^2\sqrt{u^2+a^2}}\right]_{t_{n-1}}^{t_n}$$

and

$$r < t_{n-1} - t_{m-1}$$

$$Z_2(m,n) = (\alpha_\infty^2 - 1)\left((x_1 - y_1)n_1 + (x_2 - y_2)n_2\right)\left[\frac{u}{a^2\sqrt{u^2+a^2}}\right]_{t_m+r}^{t_n} + n_3\left[\frac{1}{a^2\sqrt{u^2+a^2}}\right]_{t_m+r}^{t_n}$$

$$t_{n-1} - t_m < r < t_{n-1} - t_{m-1}$$

case 5 : $t_{m-1} + r < t_m + r < t_{n-1} < t_n$

$$Z_1(m,n) = (\alpha_\infty^2 - 1)\left((x_1 - y_1)n_1 + (x_2 - y_2)n_2\right)\left[\frac{u}{a^2\sqrt{u^2+a^2}}\right]_{t_{n-1}}^{t_n} + n_3\left[\frac{1}{a^2\sqrt{u^2+a^2}}\right]_{t_{n-1}}^{t_n}$$

and

$$r < t_{n-1} - t_{m-1}$$

$$Z_2(m,n) = (\alpha_\infty^2 - 1)\left((x_1 - y_1)n_1 + (x_2 - y_2)n_2\right)\left[\frac{u}{a^2\sqrt{u^2+a^2}}\right]_{t_{n-1}}^{t_n} + n_3\left[\frac{1}{a^2\sqrt{u^2+a^2}}\right]_{t_{n-1}}^{t_n}$$

$$r < t_{n-1} - t_m$$

case 6 : $t_{n-1} < t_{m-1} + r < t_m + r < t_n$

$$Z_1(m,n) = 0 \qquad \text{and} \qquad Z_2(m,n) = 0\,.$$

5.2 Numerical Experiment

5.2.1 Exact Solution for the Wave Equation in a Half-Space for $\Gamma = \mathbb{S}^2$ and Corresponding TDBEM solution – A Comparison

Numerical tests of the convergence behaviour for our method require the knowledge of exact solutions. In order to have suitable reference solutions for these experiments we

derive exact solutions of acoustic scattering problems in the case where the scatterer is the unit ball in $\mathbb{R}^3$.
For radial functions $u(r)$ the spherical Laplacian in $\mathbb{R}^3$ has the form:

$$\Delta u = \frac{1}{r}(ru)_{rr} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) u\,.$$

In this case, the 3D wave equation reads as

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right) u = 0\,.$$

The general solution for this equation is

$$u(t,x) = |x|^{-1}\left(\phi(|x|+t) + \psi(|x|-t)\right), \quad \text{with } |x| = r\,, \tag{5.10}$$

where ψ, ϕ are functions on $\mathbb{R}$.
We consider the Cauchy problem with radial initial data,

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \\ u|_{t=0} = u_0(|x|) \\ \frac{\partial u}{\partial t}|_{t=0} = u_1(|x|)\,, \end{cases}$$

and we extend u_0, u_1 to even functions on $\mathbb{R}$: $u_0(-r) = u_0(r)$, $u_1(-r) = u_1(r)$.
Using (5.10), we identify

$$\begin{aligned} u(0,x) &= |x|^{-1}\left(\phi(|x|) + \psi(|x|)\right) = u_0(|x|)\,, \\ \frac{\partial u}{\partial t}(0,t) &= |x|^{-1}\left(\phi'(|x|) - \psi'(|x|)\right) = u_1(|x|)\,. \end{aligned} \tag{5.11}$$

If we differentiate the first equation in (5.11) and add this to the second, we obtain

$$\begin{cases} \phi'(r) = \frac{1}{2}\left((ru_0(r))' + ru_1(r)\right) \\ \psi'(r) = \frac{1}{2}\left((ru_0(r))' - ru_1(r)\right)\,. \end{cases}$$

Now, we integrate (5.12) to get

$$\begin{cases} \phi(r) = \frac{1}{2}ru_0(r) + \frac{1}{2}\int\limits_0^r su_1(s)\,ds + C_1 \\ \psi(r) = \frac{1}{2}ru_0(r) - \frac{1}{2}\int\limits_0^r su_1(s)\,ds + C_2\,. \end{cases}$$

We use (5.10) and the initial condition to get

$$\begin{aligned} u(t,x) = \frac{1}{2r}&\left((r+t)u_0(r+t) + (r-t)u_0(r-t)\right) \\ &+ \frac{1}{2r}\int\limits_{r-t}^{r+t} su_1(s)\,ds\,. \end{aligned}$$

For a rigid half-space where $\partial_n \tilde{u} = 0$ in $\mathbb{R}^2 \times 0$, we can take

$$\tilde{u}(t,x) = u(t, r(h)) + u(t, r(-h))$$

as a solution, where $r(h) = |(x_1, x_2, x_3 - h - 1)| = \sqrt{x_1^2 + x_2^2 + (x_3 - h - 1)^2}$, $r(-h) = |(x_1, x_2, x_3 + h + 1)| = \sqrt{x_1^2 + x_2^2 + (x_3 + h + 1)^2}$ and h is the distance of the ball to the x_1x_2-plane.
Furthermore, we assume that $\frac{\partial u}{\partial t}|_{t=0} = u_1(|x|) = 0$. Then

$$\begin{aligned} u(t,x) = & \frac{1}{2r(h)} \left[(r(h)+t)u_0(r(h)+t) + (r(h)-t)u_0(r(h)-t)\right] \\ & + \frac{1}{2r(-h)} \left[(r(-h)+t)u_0(r(-h)+t) + (r(-h)-t)u_0(r(-h)-t)\right] . \end{aligned}$$

For some fixed $R > 0$, we choose

$$\begin{cases} u_0(s) = 1 + cos(\frac{\pi s}{R}) & |s| < R \\ u_0(s) = 0 & |s| \geq R \,. \end{cases}$$

For $r \geq R$, we have $u_0(t + r(h)) = 0$ and $u_0(t + r(-h)) = 0$. Hence, we get

$$\begin{aligned} u(t,x) &= \frac{r(h)-t}{2r} u_0(r(h)-t) + \frac{r(-h)-t}{2r(-h)} u_0(r(-h)-t) \\ &= \frac{r(h)-t}{2r(h)} \left[1 + \cos(\frac{\pi(r(h)-t)}{R})\right] H(R - |r(h)-t|) \\ &\quad + \frac{r(-h)-t}{2r(-h)} \left[1 + \cos(\frac{\pi(r(-h)-t)}{R})\right] H(R - |r(-h)-t|) \,, \end{aligned}$$

and its radial derivative is

$$\partial_r u = \partial_r (\frac{r(h)-t}{2r(h)} u_0(r(h)-t)) + \partial_r (\frac{r(-h)-t}{2r(-h)} u_0(r(-h)-t)) \,.$$

For the first term we obtain

$$\begin{aligned} \partial_r (\frac{r(h)-t}{2r(h)} u_0(r(h)-t)) = & \frac{t}{2r(h)^2} \left(1 + cos(\frac{\pi(t-r(h))}{R})\right) H(R - |r(h)-t|) \\ & - \frac{\pi}{R} \frac{r(h)-t}{2r(h)} \sin(\frac{\pi(r(h)-t)}{R}) H(R - |r(h)-t|) \,. \end{aligned}$$

For the second term we consider the transformation to a spherical coordinate system

$$\begin{cases} x &= r(h)\sin(\theta)\cos(\varphi) \\ y &= r(h)\sin(\theta)\sin(\varphi) \\ z &= r(h)\cos(\theta) + h + 1 \end{cases}$$

and the inverse transformation

$$\begin{cases} r(h) &= \sqrt{x^2 + y^2 + (z-h-1)^2} \\ \theta &= \arccos(\frac{z-h-1}{r(h)}) \\ \varphi &= \arctan(\frac{y}{x}) \,. \end{cases}$$

We have

$$\begin{cases} \frac{\partial x}{\partial r} & = \sin(\theta)\cos(\varphi) \\ \frac{\partial y}{\partial r} & = \sin(\theta)\sin(\varphi) \\ \frac{\partial z}{\partial r} & = \cos(\theta)\,. \end{cases}$$

The chain rule for differentiation shows

$$\begin{aligned} \partial_r F(r(-h)) &= \frac{\partial x}{\partial r} F_x(x,y,z) + \frac{\partial y}{\partial r} F_y(x,y,z) + \frac{\partial z}{\partial r} F_z(x,y,z) \\ &= \sin(\theta)\cos(\varphi) F_x + \sin(\theta)\sin(\varphi) F_y + \cos(\theta) F_z\,, \end{aligned}$$

where

$$F(r(-h)) = \frac{r(-h)-t}{2r(-h)}(1+\cos(\frac{\pi(r(-h)-t)}{R}))H(t-r(-h)+R)\,.$$

Then,

$$F_x(r(-h)) = (\partial_x r(-h))\partial_{r(-h)} F(r(-h)), \tag{5.12}$$

$$F_y(r(-h)) = (\partial_y r(-h))\partial_{r(-h)} F(r(-h)), \tag{5.13}$$

$$F_z(r(-h)) = (\partial_z r(-h))\partial_{r(-h)} F(r(-h))\,, \tag{5.14}$$

where

$$\begin{cases} \partial_x r(-h) & = \frac{x}{\sqrt{x^2+y^2+(z+h+1)^2}} \\ \partial_y r(-h) & = \frac{y}{\sqrt{x^2+y^2+(z+h+1)^2}} \\ \partial_z r(-h) & = \frac{z+h+1}{\sqrt{x^2+y^2+(z+h+1)^2}}\,. \end{cases}$$

With (5.12) it follows that

$$\begin{aligned} \partial_r F(r(-h)) = (&\sin(\theta)\cos(\varphi)(\partial_x r(-h)) + \sin(\theta)\sin(\varphi)(\partial_y r(-h)) \\ &+ \cos(\theta)(\partial_z r(-h)))\,\partial_{r(-h)} F(r(-h))\,. \end{aligned}$$

Altogether we get

$$\begin{aligned} \partial_{r(h)} u = &\left[\frac{t}{2r(h)^2}\left(1+\cos\left(\frac{\pi(r(h)-t)}{R}\right)\right) - \frac{\pi}{R}\frac{r(h)-t}{2r(h)}\sin(\frac{\pi(r(h)-t)}{R})\right] H(R-|r(h)-t|) \\ &+ \left(\left[\frac{t}{2r(-h)^2}\left(1+\cos\left(\frac{\pi(r(-h)-t)}{R}\right)\right) - \frac{\pi}{R}\frac{r(-h)-t}{2r(-h)}\sin(\frac{\pi(r(-h)-t)}{R})\right]\right. \\ &\left.\times H(R-|r(-h)-t|)\right)\frac{x^2+y^2+z^2-(h+1)^2}{r(h)r(-h)}\,. \end{aligned}$$

Using a formula of Veit [48] for the Neumann problem

$$(I-K')\varphi = 2f \tag{5.15}$$

we obtain an exact solution for the density. To do so we write $f = f_1 + f_2$ with

$$f_1 = \left[\frac{t}{2r(h)^2}\left(1+\cos\left(\frac{\pi(r(h)-t)}{R}\right)\right) - \frac{\pi}{R}\frac{r(h)-t}{2r(h)}\sin(\frac{\pi(r(h)-t)}{R})\right] H(R-|r(h)-t|)$$

and

$$f_2 = \left(\left[\frac{t}{2r(-h)^2} \left(1 + \cos\left(\frac{\pi(r(-h)-t)}{R} \right) \right) - \frac{\pi}{R} \frac{r(-h)-t}{2r(-h)} \sin(\frac{\pi(r(-h)-t)}{R}) \right] \right.$$
$$\left. \times H(R - |r(-h)-t|) \right) \frac{x^2+y^2+z^2-(h+1)^2}{r(h)r(-h)} .$$

The solution φ is then given by

$$\varphi(t) = -2 \sum_{k=0}^{\lfloor t/2 \rfloor} f_1(t-2k) + 2 \sum_{k=0}^{\lfloor t/2 \rfloor} \int_{2k}^{t} e^{-(s-2k)} f_1(t-s)\, ds .$$

We have first tested the validity and accuracy of our scheme for the Galerkin approximation

$$\langle (I-K')\varphi_{h,\Delta t}, \psi_{h,\Delta t} \rangle = 2\langle f, \psi_{h,\Delta t} \rangle$$

with piecewise constant ansatz and test functions in space and time for (5.15) on a sphere.

In Figure 5.5 the L^2-norm of the analytical and the numerical density are presented for $\Delta t = 0.025$. Figure 5.6 shows the very good agreement of the analytic and approximate solution in $x_0 = (0,0,2.8)^\top$ for $R = 0.9$, $h = 0.63$, $\Delta t = 0.1$ and 1080 uniform triangles.

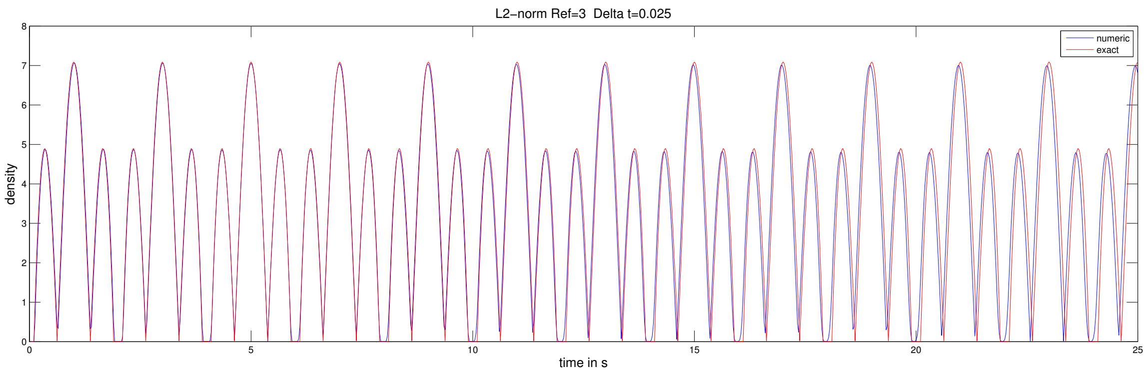


Figure 5.5: L^2-norm of the density for the exact and Galerkin solution to the integral equation $(I-K')\varphi = 2f$.

Looking at the relative discretization errors

$$\frac{\|p_{\Delta t,h}(t,x_0) - p(t,x_0)\|_{L^2([0,10])}}{\|p(t,x_0)\|_{L^2([0,10])}} \quad \text{and} \quad \frac{\|\varphi_{\Delta t,h} - \varphi\|_{L^2([0,10];L^2(\Gamma))}}{\|\varphi\|_{L^2([0,10];L^2(\Gamma))}}$$

for a family of discrete solutions where $\varphi_{\Delta t,h}$ is the TD-BE Galerkin approximation of φ and $p_{\Delta t,h} = S\varphi_{\Delta t,h}$, we obtain a convergence rate of 0.4, 0.65 resp., with respect to the degrees of freedom (dof), i.e. the product of number of time steps and number of triangles, as plotted in Figure 5.7. Here, both the time step size Δt as well as the

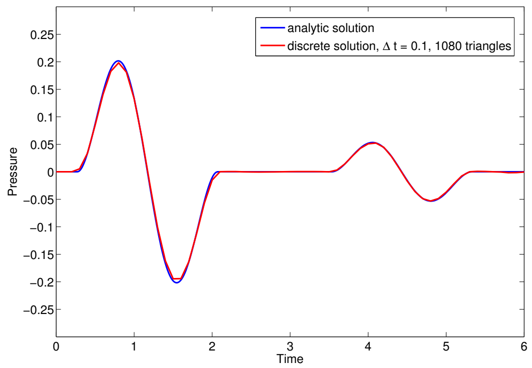


Figure 5.6: Exact sound pressure and its Galerkin approximation in $x_0 = (0, 0, 2.8)$.

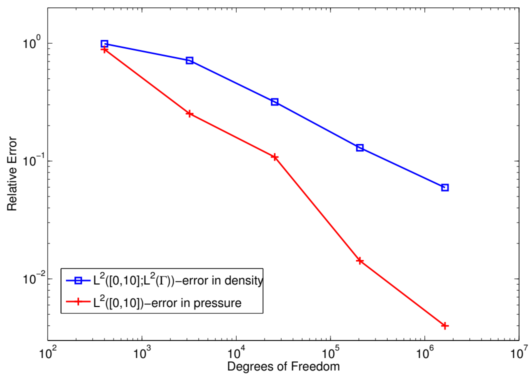


Figure 5.7: Relative L^2-errors of density $\varphi_{\Delta t,h}$ and pressure $p_{\Delta t,h}$.

mesh size h have been halved four times starting from $\Delta t = 2^{-1}$. The CFL coefficient is $\Delta t/h \approx 0.38$. See Figure 5.8 for two corresponding spacial meshes.

Note that the coerciveness of $I - K'$ in the space-time Sobolev space setting is an open question, so is the convergence of the BEM Galerkin scheme not theoretically proven yet.

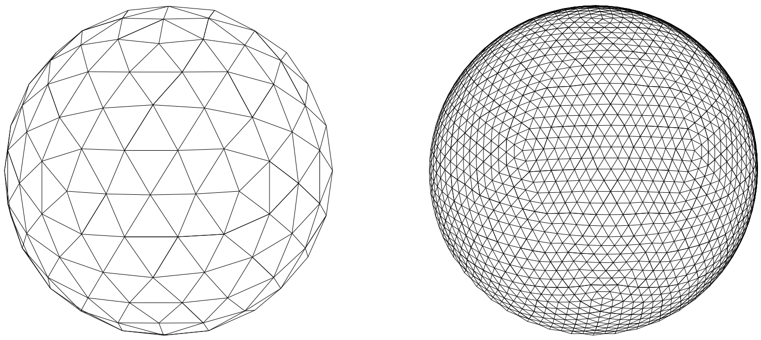


Figure 5.8: Uniform meshes for the sphere with 320 and 5120 triangles.

5.2.2 Numerical Experiments for the Sound Radiation of Tyres

We consider the radiation of time-dependent acoustic waves outside a tyre in the half-space above a rigid surface, described by the equations

$$\begin{aligned}
\frac{1}{c^2}\frac{\partial^2 p}{\partial \tau^2} - \Delta p = 0 \quad & \text{in } \mathbb{R}^+ \times \Omega^e \ , \\
p(\tau,x) = \frac{\partial p}{\partial \tau}(\tau,x) = 0 \quad & \text{in } \mathbb{R}^- \times \Omega^e \ , \\
\frac{\partial p}{\partial n}(\tau,x) = -\rho \frac{\partial^2 u_n}{\partial \tau^2} - \frac{\partial p^I}{\partial n}(\tau,x) \quad & \text{on } \mathbb{R}^+ \times \Gamma \ , \\
\frac{\partial p}{\partial x_3}(\tau,x) = 0 \quad & \text{on } \mathbb{R}^+ \times \Gamma_\infty \ .
\end{aligned}$$

Here, we use τ to denote the physical time variable in seconds, Ω^e denotes the half-space without the tyre, $p(\tau,x)$ the scattered sound pressure and $p^I(\tau,x)$ an incident wave. The coupling of the tyre vibrations and sound pressure is described by the boundary condition $\frac{\partial p}{\partial n}(\tau,x) = -\rho\frac{\partial^2 u_n}{\partial \tau^2} - \frac{\partial p^I}{\partial n}(\tau,x)$, where u describes the elastic displacement of the tyre.

For the numerical simulations, we use a rescaled time variable, $t = c\tau$, so that

$$\frac{\partial^2 p}{\partial t^2} - \Delta p = 0 \quad \text{in } \mathbb{R}^+ \times \Omega^e \ , \tag{5.16}$$

$$p(t,x) = \frac{\partial p}{\partial t}(t,x) = 0 \quad \text{in } \mathbb{R}^- \times \Omega^e \ , \tag{5.17}$$

$$\frac{\partial p}{\partial n}(t,x) = -\rho c^2 \frac{\partial^2 u_n}{\partial t^2}(t,x) - \frac{\partial p^I}{\partial n}(t,x) \quad \text{on } \mathbb{R}^+ \times \Gamma \ , \tag{5.18}$$

$$\frac{\partial p}{\partial x_3}(t,x) = 0 \quad \text{on } \mathbb{R}^+ \times \Gamma_\infty \ . \tag{5.19}$$

Our solution procedure transforms the boundary value problem into an integral equation on the boundary Γ of the tyre. It allows an efficient treatment of the half-space problem

(5.16).
We describe the sound pressure with the help of the single layer potential of the half-space,

$$p(t,x) = S\varphi(t,x) = \frac{1}{4\pi}\int_\Gamma \frac{\varphi(t-|x-y|,y)}{|x-y|}\,ds_y + \frac{1}{4\pi}\int_\Gamma \frac{\varphi(t-|x-y'|,y)}{|x-y'|}\,ds_y \tag{5.20}$$

where y' is the image of y mirrored at the plane Γ_∞.
Using (5.20) in the wave equation leads to an equivalent integral equation of the second kind:

$$(-I+K')\varphi(t,x) = 2\frac{\partial p}{\partial n}(t,x) = -2\rho c^2 \frac{\partial^2 u_n}{\partial t^2} - 2\frac{\partial p^I}{\partial n}(t,x) \tag{5.21}$$

for the density $\varphi(t,x)$ on the surface Γ. The transient adjoint double layer potential K' of the half-space as mentioned before has the form

$$\begin{aligned} K'\varphi(t,x) = {} & \frac{1}{2\pi}\int_\Gamma \frac{n_x\cdot(y-x)}{|x-y|}\left(\frac{\varphi(t-|x-y|,y)}{|x-y|^2} + \frac{\dot\varphi(t-|x-y|,y)}{|x-y|}\right) ds_y \\ & + \frac{1}{2\pi}\int_\Gamma \frac{n_x\cdot(y'-x)}{|x-y'|}\left(\frac{\varphi(t-|x-y'|,y)}{|x-y'|^2} + \frac{\dot\varphi(t-|x-y'|,y)}{|x-y'|}\right) ds_y\;. \end{aligned}$$

We solve the integral equation (5.21) in the weak sense, i.e. using a variational formulation:
Find $\varphi(t,x)$ such that for any test function $\psi(t,x)$ it holds:

$$\int\limits_{t=0}^{\infty}\int\limits_{x\in\Gamma} (-I+K')\varphi(t,x)\psi(t,x)ds_x\,dt = 2\int\limits_{t=0}^{\infty}\int\limits_{x\in\Gamma} \frac{\partial p}{\partial n}(t,x)\psi(t,x)ds_x\,dt. \tag{5.22}$$

For the numerical solution of the equation (5.22) we discretize the surface Γ of the tyre into triangles Γ_i, as well as the time interval into intervals $I_m = (t_{m-1}, t_m]$ of length Δt.
As we did before, we approximate the solution φ of the equation (5.22) by

$$\varphi_{h,\Delta t}(t,x) = \sum_{m=1}^{N_t}\sum_{i=1}^{N_s} b_i^m \gamma^m(t)\varphi_i(x) \tag{5.23}$$

where φ_i and γ^m are piecewise constant. For the test functions we choose

$$\psi_{h,\Delta t}(t,x) = \gamma^n(t)\varphi_j(x)\,.$$

The discretized formulation for (5.22) then reads as follows: Find $\varphi_{h,\Delta t}$ such that

$$\int_0^T\int_\Gamma (-I+K')\varphi_{h,\Delta t}(t,x)\gamma^n(t)\varphi_j(x)\,ds_x\,dt = 2\int_0^T\int_\Gamma \frac{\partial p}{\partial n}(t,x)\gamma^n(t)\varphi_j(x)\,ds_x\,dt \tag{5.24}$$

for $j = 1, \dots, N_s$ and $n = 1, \dots, N_t$.
According to the choice of the trial functions the integral on the left side of (5.24) turns into a sum

$$
\begin{aligned}
&-\sum_{m=1}^{N_t}\sum_{i=1}^{N_s} b_i^m \int_\Gamma \varphi_i(x)\varphi_j(x)\, ds_x \int_0^T \gamma^m(t)\gamma^n(t)\, dt\\
&+\sum_{m=1}^{N_t}\sum_{i=1}^{N_s} b_i^m \left[\int_\Gamma\int_\Gamma \left(\int_0^T \dot\gamma^m(t-|x-y|)\gamma^n(t)\, dt\right)\frac{n_x\cdot(x-y)}{2\pi|x-y|^2}\varphi_i(y)\varphi_j(x)\, ds_x\, ds_y\right.\\
&+\int_\Gamma\int_\Gamma \left(\int_0^T \gamma^m(t-|x-y|)\gamma^n(t)\, dt\right)\frac{n_x\cdot(x-y)}{2\pi|x-y|^3}\varphi_i(y)\varphi_j(x)\, ds_x\, ds_y\\
&+\int_\Gamma\int_\Gamma \left(\int_0^T \dot\gamma^m(t-|x-y'|)\gamma^n(t)\, dt\right)\frac{n_x\cdot(x-y')}{2\pi|x-y'|^2}\varphi_i(y)\varphi_j(x)\, ds_x\, ds_y\\
&\left.+\int_\Gamma\int_\Gamma \left(\int_0^T \gamma^m(t-|x-y'|)\gamma^n(t)\, dt\right)\frac{n_x\cdot(x-y')}{2\pi|x-y'|^3}\varphi_i(y)\varphi_j(x)\, ds_x\, ds_y\right]. \qquad (5.25)
\end{aligned}
$$

Note that

$$\int_0^T \gamma^m(t)\gamma^n(t)\, dt = \Delta t\, \delta_{nm}\,, \qquad (5.26)$$

$$\int_0^T \dot\gamma^m(t-|x-y|)\gamma^n(t)\, dt = -\chi_{E_{n-m-1}}(x-y) + \chi_{E_{n-m}}(x-y)\,, \qquad (5.27)$$

$$
\begin{aligned}
\int_0^T \gamma^m(t-|x-y|)\gamma^n(t)\, dt &= (t_{n-m+1} - |x-y|) - \chi_{E_{n-m}}(x-y)\\
&\quad + (|x-y| - t_{n-m-1})\chi_{E_{n-m-1}}(x-y)\,. \qquad (5.28)
\end{aligned}
$$

Herewith we obtain for (5.25)

$$-\Delta t \sum_{i=1}^{N_s} b_i^n \int_\Gamma \varphi_i(x)\varphi_j(x)\, ds_x + \sum_{m=1}^{N_t}\sum_{i=1}^{N_s} b_i^m \left(A_{im} + A'_{im}\right)\,, \qquad (5.29)$$

where

$$
\begin{aligned}
A_{im} &:= -\int_{E_{n-m-1}} \frac{n_x\cdot(x-y)}{2\pi|x-y|^2}\varphi_i(y)\varphi_j(x)\, ds_x ds_y + \int_{E_{n-m}} \frac{n_x\cdot(x-y)}{2\pi|x-y|^2}\varphi_i(y)\varphi_j(x) ds_x ds_y\\
&+ t_{n-m+1}\int_{E_{n-m}} \frac{n_x\cdot(x-y)}{2\pi|x-y|^3}\varphi_i(y)\varphi_j(x) ds_x ds_y - \int_{E_{n-m}} \frac{n_x\cdot(x-y)}{2\pi|x-y|^2}\varphi_i(y)\varphi_j(x) ds_x ds_y\\
&+ \int_{E_{n-m-1}} \frac{n_x\cdot(x-y)}{2\pi|x-y|^2}\varphi_i(y)\varphi_j(x)\, ds_x ds_y - t_{n-m-1}\int_{E_{n-m-1}} \frac{n_x\cdot(x-y)}{2\pi|x-y|^3}\varphi_i(y)\varphi_j(x) ds_x ds_y\,. \qquad (5.30)
\end{aligned}
$$

A'_{im} is obtained by taking y' instead of y and E'_{n-m}, E'_{n-m-1} instead of E_{n-m}, E_{n-m-1}.

Therefore (5.25) becomes

$$
\begin{aligned}
&- \Delta t \sum_{i=1}^{N_s} b_i^m \int_\Gamma \varphi_i(x)\varphi_j(x)\, ds_x + \sum_{m=1}^{N_t}\sum_{i=1}^{N_s} b_i^m \Bigg[t_{n-m+1} \Bigg(\int_{E_{n-m}} \frac{n_x \cdot (x-y)}{2\pi |x-y|^3} \varphi_i(y)\varphi_j(x) ds_x ds_y \\
&+ \int_{E'_{n-m}} \frac{n_x \cdot (x-y')}{2\pi |x-y'|^3} \varphi_i(y)\varphi_j(x) ds_x ds_y \Bigg) - t_{n-m-1} \Bigg(\int_{E_{n-m-1}} \frac{n_x \cdot (x-y)}{2\pi |x-y|^3} \varphi_i(y)\varphi_j(x) ds_x ds_y \\
&+ \int_{E'_{n-m-1}} \frac{n_x \cdot (x-y')}{2\pi |x-y'|^3} \varphi_i(y)\varphi_j(x) ds_x ds_y \Bigg) \Bigg] \,. \qquad (5.31)
\end{aligned}
$$

Hence we can write (5.25) in matrix form as

$$
-\Delta t \; I \; \varphi^n + \sum_{m=1}^{n} (K')^{n-m} \; \varphi^m \qquad (5.32)
$$

with the vector $\varphi^n = (b_1^n, \ldots, b_{N_s}^n)^T$ and

$$
\begin{aligned}
(K')^l := t_{l+1} &\left(\int_{E_l} \frac{n_x \cdot (x-y)}{2\pi |x-y|^3} \varphi_i(y)\varphi_j(x) ds_x ds_y + \int_{E'_l} \frac{n_x \cdot (x-y')}{2\pi |x-y'|^3} \varphi_i(y)\varphi_j(x) ds_x ds_y \right) \\
- t_{l-1} &\left(\int_{E_{l-1}} \frac{n_x \cdot (x-y)}{2\pi |x-y|^3} \varphi_i(y)\varphi_j(x) ds_x ds_y + \int_{E'_{l-1}} \frac{n_x \cdot (x-y')}{2\pi |x-y'|^3} \varphi_i(y)\varphi_j(x) ds_x ds_y \right) .
\end{aligned}
\qquad (5.33)
$$

For the right hand side F^n in (5.24) we have

$$
\begin{aligned}
\int_0^T \int_\Gamma -\rho c \dot{v}_n(t,x) \psi_{h,\Delta t} \; ds_x \; dt = -\rho c &\int_\Gamma \int_{t_{n-1}}^{t_n} \dot{v}_n(t,x) \; dt \varphi_i(x) \; ds_x \\
&- \int_\Gamma \int_{t_{n-1}}^{t_n} \frac{\partial p^I}{\partial n}(t,x) \; dt \varphi_i(x) \; ds_x \qquad (5.34)
\end{aligned}
$$

with $v_n = \dot{u}_n$.
Hence $F^n = f_i^n$ with

$$
\begin{aligned}
f_i^n = -\rho c &\left[\int_\Gamma v_n(t_n, x)\varphi_i(x) \; ds_x - \int_\Gamma v_n(t_{n-1}, x)\varphi_i(x) \; ds_x \right] \qquad (5.35) \\
&- \int_\Gamma \int_{t_{n-1}}^{t_n} \frac{\partial p^I}{\partial n}(t,x) \; dt \varphi_i(x) \; ds_x \qquad (5.36)
\end{aligned}
$$

The Galerkin discretization in space and time leads to a *marching in on-time* (MOT) algorithm, as seen in Chapter 2,

$$
(-\Delta t \; I + (K')^0)\varphi^n = 2F^n - \sum_{m=1}^{n-1} (K')^{n-m} \varphi^m \; .
$$

5.2.3 Numerical Experiment for a Vibrating Tyre

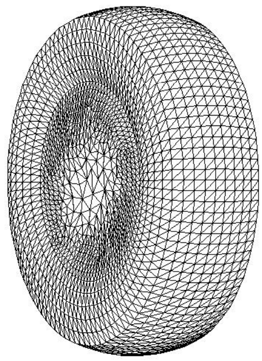


Figure 5.9: Discretization of tyre used for computation of tyre vibration and Horn effect

We validate our code by comparing its results with results in frequency domain obtained by W. Kropp and O. von Estorff and their groups at Chalmers University, Gothenburg resp. TU Hamburg-Harburg. We solve (5.19) with $p^I = 0$ using the MOT scheme (5.23), for the normal velocities $\dot{u}_N$ obtained for a model tyre.
In practice, we use data for $\dot{u}_N = v_N$ in frequency domain provided by the colleagues above and Fourier transform it into the time-domain. Kropp and von Estorff solve the Helmholtz equation

$$\Delta\hat{p} + \omega^2\hat{p} = 0 \qquad \text{in } \mathbb{R}^+ \times \Omega \tag{5.37a}$$

$$\frac{\partial\hat{p}}{\partial n}(\omega, x) = -\rho i\omega v_n(\omega, x) \qquad \text{on } \mathbb{R}^+ \times \Gamma \tag{5.37b}$$

$$\frac{\partial\hat{p}}{\partial x_3}(\omega, x) = 0 \qquad \text{on } \mathbb{R}^+ \times \Gamma_\infty \tag{5.37c}$$

They use piecewise constant ansatz functions and collocation together with a Fast Multipole Method [27] to solve

$$\frac{1}{2}\hat{p}_i + \sum_{j=1}^{N}\left(\int_{\triangle_j}\frac{\partial G(x,y)}{\partial n_y}\,ds_y\right)\hat{p}_j = \sum_{j=1}^{N}\int_{\triangle_j} G(x,y)\,ds_y\rho i\omega v_N(\omega, y_j) \tag{5.38}$$

with

$$G(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|} + \frac{e^{ik|x-y'|}}{4\pi|x-y'|}\,. \tag{5.39}$$

Here y_j is the midpoint of $\triangle_j$ and $\hat{p}_i$ is the numerical approximation to $\hat{p}$ in $\triangle_j$. While this formulation in practice yields reliable results up to 1000Hz, for larger frequencies they use a Burton-Miller stabilized method.

In the model, we consider an idealized tyre without profile (205/55R16). The corresponding mesh consists of 12044 elements with 6027 nodes, see Figure 5.9.

The velocities $\dot{u}_N$ are given at each node as a function of $f = \frac{\omega}{2\pi}$, at 513 equidistant frequencies from $f = 0$ to $f = 1809.4$ Hz. They are then Fourier transformed into a function of time, wich is used to calculate the right hand side.

After solving (5.24) for the density φ with $\Delta t = 3.125.10^{-4}$ s, we evaluate the sound pressure $p(t, x)$ in 320 points in the hemisphere $\{x \in \mathbb{R}^3_+ : \|x\|_2 = 1\}$ outside the tyre.

The resulting $p(t, x)$ for 320 resp. 200 time steps is Fourier transformed into the frequency domain for each of the 320 points.

The A-weighted sound pressure level is a recognized approximates the human's perception of noise [26]. Figure 5.10 shows the A-weighted sound pressure level of the acoustic wave radiated from the tyre averaged over 321 points in the hemisphere $\{x \in \mathbb{R}^3_+ : \|x\|_2 = 1\}$. As a reference, we use the results of [27], blue curve in Figure 5.10, which was calculated by a BEM collocation method with piecewise constant trail functions [27] for the Helmholtz equation.

It is worth pointing out that for the blue curve a Burton-Miller stabilization is used for frequencies above 1000Hz. The remaining curves all result form the same TDBEM simulation where the difference is only the starting time from which onwards the point evaluation of the sound pressure is considered. Except for the black curve, the other once have the same qualitative behavior and only differ significantly to the blue curve for the third-octave bands with center frequency 1600Hz and 2000Hz. Due to the use of the particle velocities from the frequency domain and the discretization of the ramp up function, a shock wave is emitted at the beginning of the simulation. Considering this artificial wave also in the Fourier transformation leads to distorted amplitudes in the frequency spectrum, hence the black curve. This can be avoided by using the point evaluation of the sound pressure from time t_j onwards for the discrete Fourier transformation.

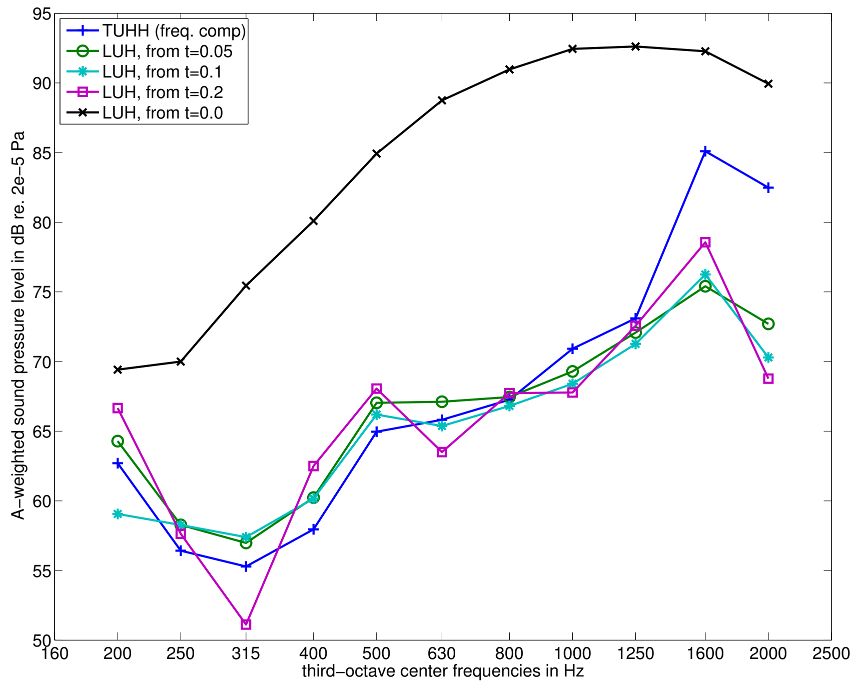


Figure 5.10: Comparison of the A-weighted sound pressure level averaged over 320 points and frequency bands for TDBEM and frequency–domain BEM, see text for details.

Remark 5.1. *The calculated pressure by the BEM-model in the time domain is transformed by the FFT into the frequency domain. The frequency range is determined by the step size of the time discretization and ranges from* 0 *to* $\frac{1}{\Delta t}$. *With* N_t *time steps we obtain a frequency resolution of* $\frac{1}{(N_t+1)\Delta t}$.

5.2.4 Numerical Experiment for the Horn Effect

The horn effect is the amplification of the sound field radiated by sound sources close to the contact area between the tyre and the flat ground. It is due to the horn-like geometry between tyre and street.
Experimentally, the horn effect is observed by measuring the pressure field radiated by the tyre due to a given excitation on the one hand and the pressure radiated by the same tyre and the same excitation but in the presence of a road on the other hand. By taking the ratio of both pressure fields, amplification factors are calculated which quantify the horn effect for the given tyre and the given excitation.
Because of the principle of acoustic reciprocity the measured pressure field is the same if the source and the microphone switch positions. This means that the noise source

can be placed in front of the tyre and that the microphone can be placed inside the contact region. This usually results in simpler experimental set-ups due to the fact that the noise source, which usually requires a bit of space, is placed away from the contact zone. This type of arrangement is used in this work.
Our calculations allow us to quantify the horn effect for a given tyre.
We consider the idealized last tyre from Section 5.2.3. It is located at a distance h_t above the origin $(0,0,0)$.
Engineers have studied the horn effect for a monopole point source located in $y_{src} \in \Gamma_\infty$ of strength $1N/m$ for various single frequencies. They consider Neumann data given for $y_{src} = (0,0,d_s)$ by $\frac{\partial p^I}{\partial n} = \frac{\partial G(\omega,x,y_{src})}{\partial n}$ with

$$G(\omega, x, y_{src}) = \frac{e^{i\omega|x-y_{src}|}}{4\pi|x-y_{src}|} + \frac{e^{i\omega|x-y'_{src}|}}{4\pi|x-y'_{src}|}.$$

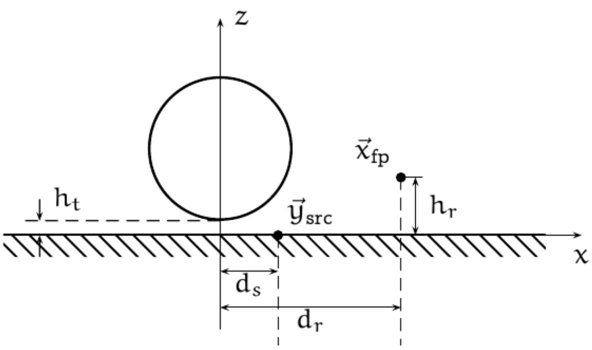


Figure 5.11: Horn geometry

In the time domain, after Fourier transformation, we obtain a time-dependent point source

$$G(t, x, y_{src}) = \frac{\delta(t-|x-y_{src}|)}{4\pi|x-y_{src}|} + \frac{\delta(t-|x-y'_{src}|)}{4\pi|x_s-y'_{src}|}. \tag{5.40}$$

Note that $\frac{\partial G(t,x,y_{src})}{\partial x_3} = 0$ on Γ_∞.
Neglecting tyre vibrations $\rho c^2 \frac{\partial^2 u_n}{\partial t^2}$ we have to solve the integral equation

$$(-I+K')\varphi(t,x) = -2\frac{\partial G}{\partial n}(t,x) = f(t,x), \qquad (t,x) \in \mathbb{R}^+ \times \Gamma. \tag{5.41}$$

We use the additional parameters $d_s = 80mm, \quad h_t = 10mm, \quad d_r = 1m, \quad h_r = 0m$ (see Figure 5.11).
The components of the vector on the right hand side can be calculated analytically.

Since we consider the case $y_{src} = y'_{src}$ we have

$$\langle f, \varphi_j \chi_{I_n} \rangle = -4 \int_{I_n} dt \int_\Gamma ds_x \left\{ \frac{n_x \cdot (y_{src} - x)}{4\pi |x - y_{src}|^2} \dot{\delta}(t - |x - y_{src}|) \right. \tag{5.42}$$

$$\left. + \frac{n_x \cdot (y_{src} - x)}{4\pi |x - y_{src}|^3} \delta(t - |x - y_{src}|) \right\} \varphi_j(x)$$

$$= (I) + (II) \,. \tag{5.43}$$

Here

$$\begin{aligned} (II) &= - \int_{\Gamma \cap \{x \in \Gamma | \; t_{n-1} \leq |x - y_{src}| \leq t_n\}} ds_x \frac{n_x \cdot (y_{src} - x)}{\pi |x - y_{src}|^3} \varphi_j(x) \\ &= - \int_{T_j \cap E(y_{src})} ds_x \frac{n_x \cdot (y_{src} - x)}{\pi |x - y_{src}|^3} \varphi_j(x) \end{aligned}$$

is an integral over the domain of influence of y_{src}. We compute it as in [40, (4.7)],[33]. The first term (I) can be evaluated more explicitly.

$$\begin{aligned} (I) &= 4 \int_{\mathbb{R}} dt \int_\Gamma ds_x \frac{n_x \cdot (y_{src} - x)}{4\pi |x - y_{src}|^2} \delta(t - |x - y_{src}|) \{ -\delta_{t_n} + \delta_{t_{n-1}} \} \varphi_j(x) \\ &= \int_{T_j} ds_x \frac{n_x \cdot (y_{src} - x)}{\pi |x - y_{src}|^2} \{ -\delta(t_n - |x - y_{src}|) + \delta(t_{n-1} - |x - y_{src}|) \} \\ &= \frac{1}{\pi} \int_{T_j \cap \{|x - y_{src}| = t_{n-1}\}} \frac{n_x \cdot (y_{src} - x)}{|x - y_{src}|^2} ds_x - \frac{1}{\pi} \int_{T_j \cap \{|x - y_{src}| = t_n\}} \frac{n_x \cdot (y_{src} - x)}{|x - y_{src}|^2} ds_x \\ &= \frac{1}{\pi t_{n-1}^2} \int_{T_j \cap \{|x - y_{src}| = t_{n-1}\}} n_x \cdot (y_{src} - x) \, ds_x - \frac{1}{\pi t_n^2} \int_{T_j \cap \{|x - y_{src}| = t_n\}} n_x \cdot (y_{src} - x) \, ds_x \\ &= n_x \cdot (y_{src} - x) \left\{ \frac{\zeta(t_{n-1})}{\pi t_{n-1}^2} - \frac{\zeta(t_n)}{\pi t_n^2} \right\} \end{aligned} \tag{5.44}$$

where $\zeta(t)$ denotes the length of the curve $T_j \cap \{|x - y_{src}| = t\}$.
Here we note that the integrand $n_x \cdot (y_{src} - x)$ is constant on each triangle T_j. Indeed, if $a, b \in T_j$, then $a - b \perp n_x$ and therefore

$$n_x \cdot (y_{src} - x) = n_x \cdot (y_{src} - y - (x - y)) = n_x \cdot (y_{src} - y) \,.$$

The amplification due to the horn effect in x_{fp} is given by

$$\Delta L_H(\omega) = 20 \log_{10} \left(\frac{|\hat{p}_1(\omega, x_{fp})|}{|\hat{p}_2(\omega, x_{fp})|)} \right) ,$$

where $\hat{p}_2$ is the Fourier transformed of the emitted Dirac impulse and $\hat{p}_1$ the Fourier transformed of the Dirac impulse overlayed with the sound ration of the tyre. The Fourier transformations are realized by the discrete FFT applied to a sample of each wave where the time step size is the same has for the computation of the density. For the sampling of the Dirac impulse the function $\delta(t - |x_{fp} - y_{src}|)$ is first approximated by a rectangular function where the rectangular has a width of Δt, center $|x_{fp} - y_{scr}|$ and height $(\Delta t)^{-1}$.

Calculation in the time domain

We use the discretization of Section 5.2.3. As time steps we take $\Delta t = 0.01$, $\Delta t = 0.04$, $\Delta t = 0.16$ which corresponding to $2.9155e-05\,s$, $1.1662e-04\,s$, $4.6647e-04\,s$, and we compute until $T = 24$. We solve the second kind integral equation (5.41) with right hand side (5.42). The MOT method provides the coefficients $\varphi^m = (b_i^m)_i$ at the times $m\Delta t$ and thus the discretized density (5.23).
Substituting (5.23) into (5.20) makes the calculation of the sound pressure $p(t, x_{fp})$ at the point x_{fp} possible, which gives $p_1 = p + G$, see (5.40).

Comparison between measurements and calculations of the Horn Effect

The following figure shows experimentally measured amplification curves for 4 different microphone positions in the frequency range from 300 Hz to 6 kHz. These measurements serve as validation data for the BEM solutions.
When the distance from the horn is reduced the maximum of the amplification is reached and is shifted to lower frequencies, [10].

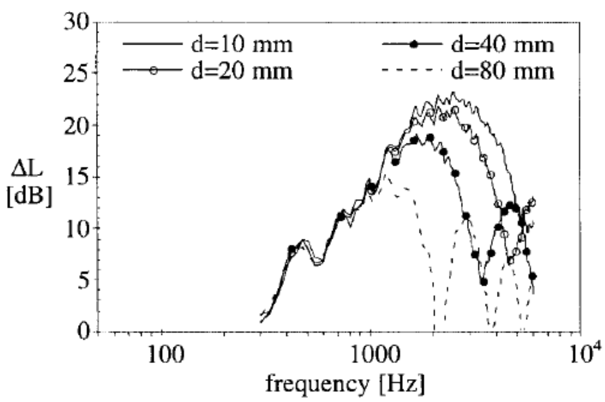


Figure 5.12: Experimentally measured amplification due to the Horn effect for different values of d

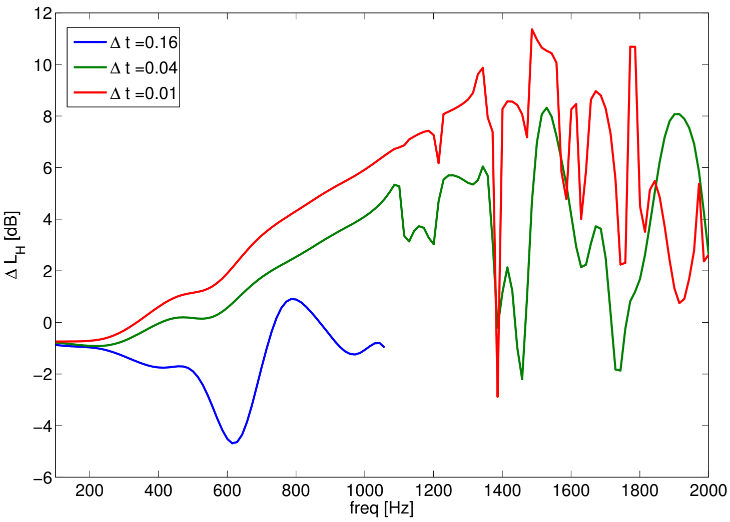


Figure 5.13: Calculated amplification due to the Horn effect for $d = 80mm$, tyre $1\,mm$ above ground.

Figure 5.13 displays the computed amplifications for the frequencies from 200 to 2000 Hz. Calculations with the model and parameters described above are in qualitative agreement with the experimental values. Especially they allow to predict the maximal amplification and the frequencies at which it occurs.

6 Rolling Tyre

External noise radiating from a tyre is one of the dominant noise sources of a vehicle as shown in Figure 6.1. For moderate speeds it is the main noise heard by an an observer at a distance from the street.
This chapter concerns the noise generated by a tyre rolling on the road. For this purpose we develop a 3D time domain boundary element method based on the fundamental solution for a moving wave equation with rotating data on the tyre.

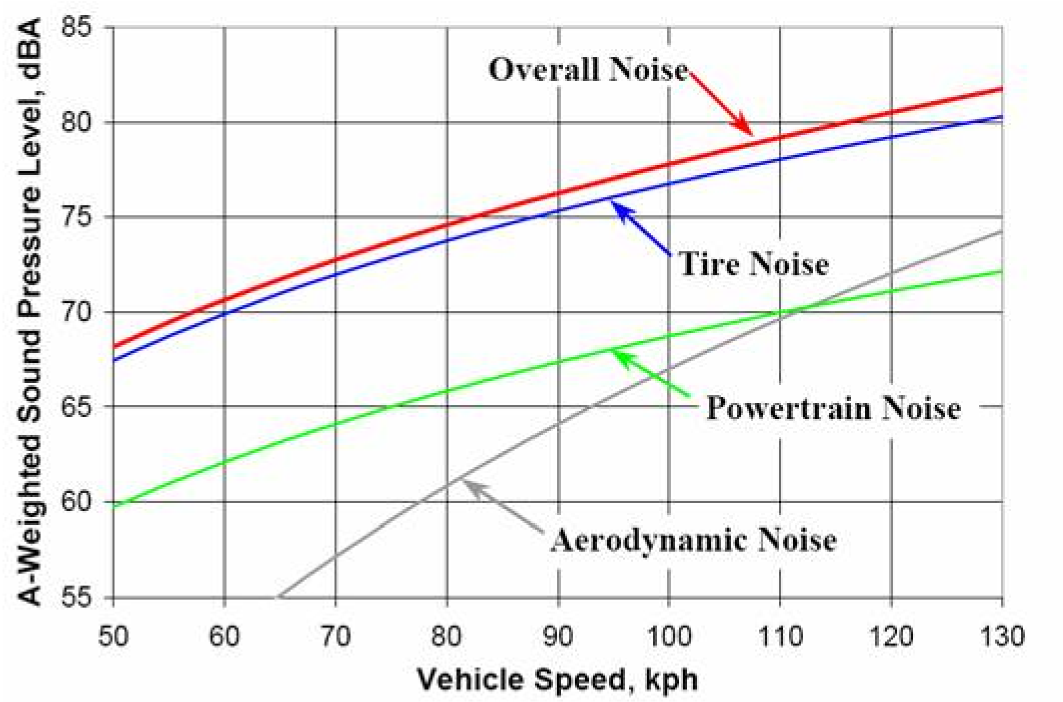


Figure 6.1: Noise sources of a driving vehicle, Source: R. Bernard, R. Wayson. An Introduction to Tire–Pavement Noise of Asphalt Pavement, Purdue University, 2005

6.1 Green's Function for the Moving Wave Equation

We consider an acoustic point source $q(t, x)$ moving with constant speed v along the x_1-axis at height h above the x_1x_2-plane. The acoustic sound pressure p is measured

at $y = (y_1, y_2, y_3)$.
The Lorentz transformation with respect to x and t is

$$\begin{aligned}\tilde{t} &= \gamma(t - Mx)\\ \tilde{x}_1 &= \gamma(x_1 - Mt)\\ \tilde{x}_2 &= x_2\\ \tilde{x}_3 &= x_3\,,\end{aligned}$$

where M is the Mach number, $M = v$ if $c = 1$, and $\gamma = \frac{1}{\sqrt{1-M^2}}$.
This transformation will be applied to the wave equation (2.1) leading to a new equation

$$\Box\, u_L := \frac{1}{c^2}\frac{\partial^2 u_L}{\partial \tilde{t}^2} - \Delta_L u_L = 0\,. \tag{6.1}$$

u_L is the solution of (6.1) and Δ_L is the Laplace operator written in the new coordinates $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$.
The Green's function, which describes a moving point source in the new coordinates, is given by

$$G(\tilde{x}, \tilde{t}; \tilde{y}, \tilde{\tau}) = \frac{\delta(\tilde{t} - \tilde{\tau} - \tilde{R})}{4\pi\tilde{R}[1 - M_{\tilde{R}}]}\,,$$

where $M_{\tilde{R}} = \frac{M\tilde{\mathbf{R}}}{\tilde{R}}$, $\tilde{\mathbf{R}} = \tilde{x} - \tilde{y}$ and $\tilde{R} = |\tilde{x} - \tilde{y}|$.
The Lorentz transformation allows the moving source to be treated as a stationary source in the Lorentz frame, and provides a model for the motion of the source.
The Dirichlet or Neumann boundary conditions are preserved under Lorentz transformation and we obtain the corresponding half-space Green's function, for example

$$G(\tilde{x}, \tilde{t}; \tilde{y}, \tilde{\tau}) = \frac{\delta(\tilde{t} - \tilde{\tau} - \tilde{R})}{4\pi\tilde{R}[1 - M_{\tilde{R}}]} + \frac{\delta(\tilde{t} - \tilde{\tau} - \tilde{R}')}{4\pi\tilde{R}'[1 - M_{\tilde{R}'}]}$$

for Neumann boundary conditions on Γ_∞.
Using some unknown density φ, we can represent the pressure as follows:

$$p(\tilde{t}, \tilde{x}) = S\varphi(\tilde{t}, \tilde{x}) = \int_\Gamma \frac{\varphi(\tilde{t} - \tilde{R}, y)}{4\pi\tilde{R}[1 - M_{\tilde{R}}]}\, ds_y + \int_\Gamma \frac{\varphi(\tilde{t} - \tilde{R}', y)}{4\pi\tilde{R}'[1 - M_{\tilde{R}'}]}\, ds_y\,. \tag{6.2}$$

We obtain the time-domain boundary integral equation

$$(-I + K')\varphi = 2\frac{\partial f}{\partial n} \tag{6.3}$$

and its discretized variant

$$\langle(-I + K')\varphi_{h,\Delta t}, \psi_{h,\Delta t}\rangle = 2\langle\frac{\partial f}{\partial n}, \psi_{h,\Delta t}\rangle\,. \tag{6.4}$$

Here the time-domain adjoint double layer operator in the moving frame is given by

$$\begin{aligned}K'\varphi(\tilde{t}, \tilde{x}) = \frac{1}{2\pi}\int_\Gamma &-\frac{n_x\cdot\tilde{\mathbf{R}}}{\tilde{R}[1 - M_{\tilde{R}}]}\dot{\varphi}(\tilde{t} - \tilde{\mathbf{R}}, y)\, ds_y - \frac{n_x\cdot\tilde{\mathbf{R}} + \tilde{R}n_x\cdot M}{\tilde{R}[1 - M_{\tilde{R}}]^2}\varphi(\tilde{t} - \tilde{\mathbf{R}}, y)\, ds_y\\ + \frac{1}{2\pi}\int_\Gamma &-\frac{n_x\cdot\tilde{\mathbf{R}}'}{\tilde{R}'[1 - M_{\tilde{R}}]}\dot{\varphi}(\tilde{t} - \tilde{\mathbf{R}}', y)\, ds_y - \frac{n_x\cdot\tilde{\mathbf{R}}' + \tilde{R}'n_x\cdot M}{\tilde{R}'[1 - M_{\tilde{R}}]^2}\varphi(\tilde{t} - \tilde{\mathbf{R}}, y)\, ds_y\,.\end{aligned}$$

6.2 Exact Solution for the Moving Wave Equation

We consider the exact solution of the wave equation in Section 5.2.1. After a Lorentz transformation the solution of the wave equation in the new coordinate system is

$$\begin{aligned} U_{rl}(t,x_1,x_2,x_3) = \frac{r(h)-\gamma(t-vx_1)}{2r(h)}U_0(r(h)-\gamma(t-vx_1)) \\ + \frac{r(-h)-\gamma(t-vx_1)}{2r(-h)}U_0(r(-h)-\gamma(t-vx_1)) \end{aligned}$$

where

$$\begin{aligned} r(h) &= \sqrt{\tilde{x}_1^2+\tilde{x}_2^2+(\tilde{x}_3-h)^2} = \sqrt{\gamma^2(x_1-vt)^2+x_2^2+(x_3-h)^2} \\ r(-h) &= \sqrt{\tilde{x}_1^2+\tilde{x}_2^2+(\tilde{x}_3+h)^2} = \sqrt{\gamma^2(x_1-vt)^2+x_2^2+(x_3+h)^2}\,. \end{aligned}$$

The solution $U_{rl}(t,x_1,x_2,x_3)$ describes a noise source that moves with velocity v in the x_1-direction with $\frac{\partial U_{rl}}{\partial x_3}=0$ on the road.
The solution for an observer sitting under the tyre on the road is

$$\begin{aligned} U(\tilde{t},\tilde{x}_1,\tilde{x}_2,\tilde{x}_3) = {} & \frac{\tilde{r}-\gamma(\tilde{t}-\frac{v}{c^2}(\tilde{x}_1+v\tilde{t}))}{2\tilde{r}}U_0(\tilde{r}-\gamma(t-\frac{v}{c^2}x_1)) \\ & + \frac{\tilde{r}'-\gamma(\tilde{t}-\frac{v}{c^2}(\tilde{x}_1+v\tilde{t}))}{2\tilde{r}'}U_0(r'-\gamma(\tilde{t}-\frac{v}{c^2}(\tilde{x}_1+v\tilde{t}))) \end{aligned}$$

where

$$\tilde{r}(h) = \sqrt{\gamma^2\tilde{x}_1^2+\tilde{x}_2^2+(\tilde{x}_3-h)^2}, \qquad \tilde{r}(-h) = \sqrt{\gamma^2\tilde{x}_1^2+\tilde{x}_2^2+(\tilde{x}_3+h)^2}\,.$$

Therefore

$$\begin{aligned} U(\tilde{t},\tilde{x}_1,\tilde{x}_2,\tilde{x}_3) = {} & \frac{\tilde{r}-\frac{\tilde{t}}{\gamma}+(\gamma-\frac{1}{\gamma})\tilde{x}_1}{2\tilde{r}}U_0(\tilde{r}-\frac{\tilde{t}}{\gamma}+(\gamma-\frac{1}{\gamma})\tilde{x}_1) \\ & + \frac{\tilde{r}'-\frac{\tilde{t}}{\gamma}+(\gamma-\frac{1}{\gamma})\tilde{x}_1}{2\tilde{r}'}U_0(\tilde{r}'-\frac{\tilde{t}}{\gamma}+(\gamma-\frac{1}{\gamma})\tilde{x}_1) \end{aligned}$$

This is the solution of the wave equation moving on the road.
We will also be interested in rolling tyres, which rotate in addition to moving. They are simply described by rotating data f on the moving tyre.
Given data $f = f(t,x_1,x_2,x_3)$ in the stationary frame we get data for the rotating frame at angular velocity ω around the x_2-axis through $x_3 = h$:

$$\begin{pmatrix} \tilde{x}_1+\omega Rt \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} + \begin{pmatrix} \cos(\omega t) & 0 & -\sin(\omega t) \\ 0 & 1 & 0 \\ \sin(\omega t) & 0 & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3-h \end{pmatrix}.$$

Solving for (x_1,x_2,x_3) gives

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3-h \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & 0 & \sin(\omega t) \\ 0 & 1 & 0 \\ -\sin(\omega t) & 0 & \cos(\omega t) \end{pmatrix} \begin{pmatrix} \tilde{x}_1+\omega Rt \\ \tilde{x}_2 \\ \tilde{x}_3-h \end{pmatrix},$$

so that we get the data on the rolling tyre:

$$F_{roll}(t, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = F(t, \cos(\omega t)\tilde{x}_1 + \sin(\omega t)(\tilde{x}_3 - h) + \omega R t, x_2, h - \sin(\omega t)\tilde{x}_1 + \cos(\omega t)(\tilde{x}_3 - h))\,.$$

The angular velocity ω and the velocity v in the x_1 direction are related by the condition $v = \omega R$.

6.3 Numerical Experiment

In this section we present the numerical results for some test cases for the outgoing sound wave in a moving frame. Here, the performance of the algorithm is demonstrated by several examples for moving or rolling spheres and tyres.
In all examples we solve the Galerkin discretization (6.4) of the integral equation (6.3), using piecewise constant ansatz and test functions in space and time.

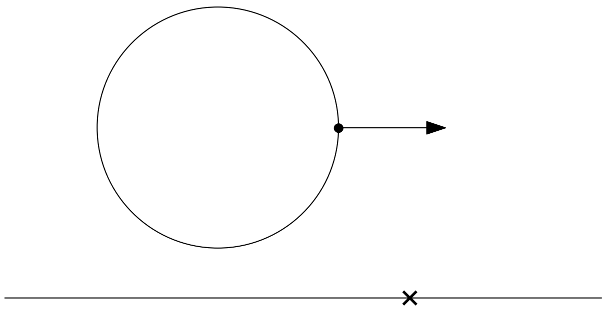


Figure 6.2: Geometry of the moving tyre

For the first example we choose a sound source at the front of the sphere, which moves with constant velocity $v = 24,69km/h = 6.8583m/s$ above the plane ($h = 0.001m$) and radiates with a frequency $frq = 54.59$ Hz.
The Cauchy data on the surface of the sphere are

$$f(t,x) = \frac{cos(frq(t - \frac{|x-y_{src}|}{c}))}{4\pi(|x - y_{src}| - (x - y_{src}).M)} + \frac{cos(frq(t - \frac{|x-y'_{src}|}{c}))}{4\pi(|x - y'_{src}| - (x - y'_{src}).M)}. \tag{6.5}$$

We approximate the sphere by an icosahedron of 320 elements. In order to satisfy an appropriate relation $\beta = \frac{\tilde{c}\Delta t}{\Delta x} = 0.5$ between time and spatial discretizations, a time step $\Delta t = 0.4908$ which corresponds to $0.001431s$ should be chosen. The sound pressure is evaluated at position $(2, 0, 0)$.

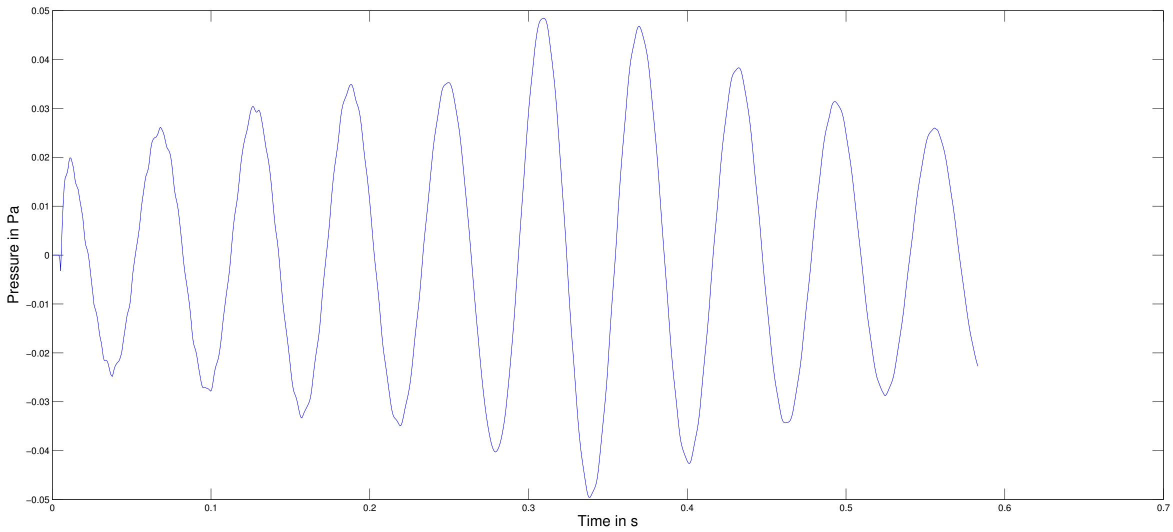


Figure 6.3: Sound pressure in point $(2, 0, 0)$ as a function of time (in seconds).

Figure 6.3 shows how the noise level increases as the source approaches the microphone and decreases when the source has passed.
In this example our resolution is too small to observe the Doppler shift $\Delta frq = \frac{v}{c-v} frq$ of the frequency for a moving source source.
To see this effect in the numerical solution, we choose $\Delta t = 0.15$ which corresponds to $0.000437s$ and perform 2194 time steps, the other parameters being as in the previous experiment. We evaluate the sound pressure as a function of time in the point $x_0 = (13, 0, 0)$. Figure 6.4 shows the Fourier transformed sound pressure $p(f, x_0)$ (normalized to amplitude 1) in x_0 for frequencies up to 100Hz and compares the signal with the Fourier transform of a sinusoidal wave $\sin(t_j)$ in the same nodes $t_j = (j-1)\Delta t$. We note the shift to the right of the dominant frequency, and with the current parameters our frequency resolution suffices to clearly distinguish the two peaks. Figure 6.5 zooms into the frequency band from 50 to 60Hz, where the location of the peaks is at the expected frequencies 54.59Hz and $\frac{6.8583}{343-6.8583} \times 54.59 + 54.59 = 55.7$Hz.

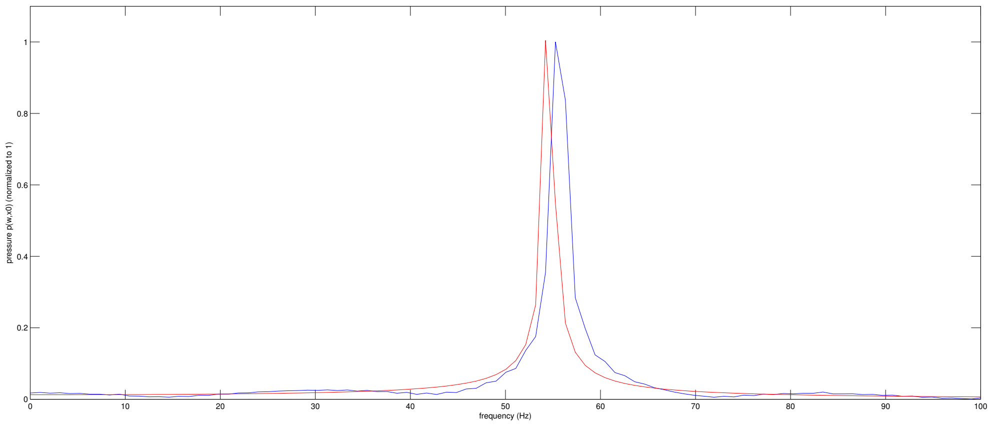


Figure 6.4: Doppler effect: Fourier transformed sound pressure vs. sinusoidal signal as a function of frequency in the point $x_0 = (13, 0, 0)$.

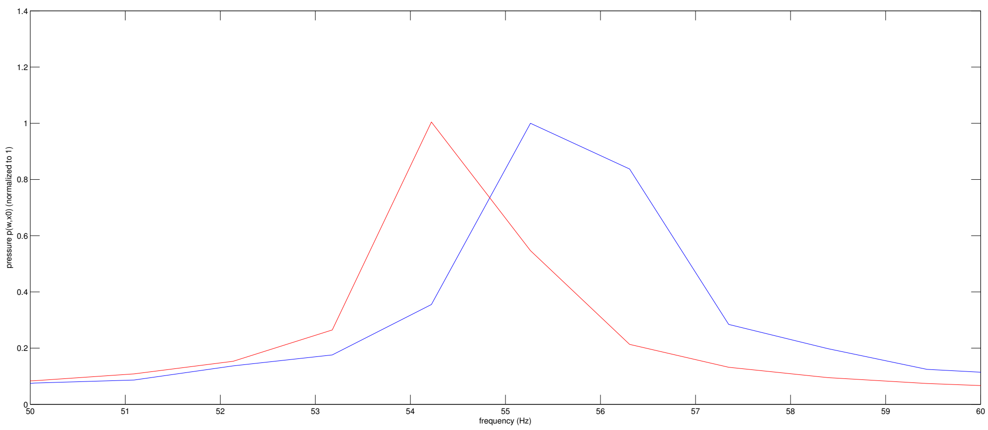


Figure 6.5: Doppler effect: Detailed view of the frequency band from 50 to 60Hz.

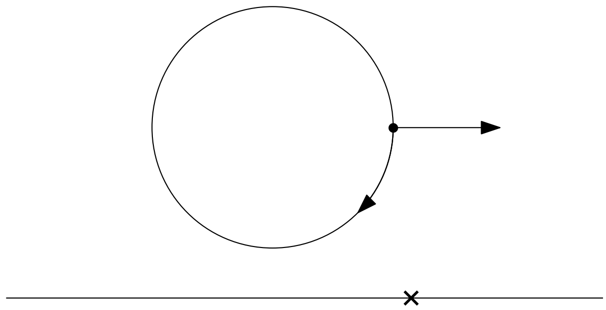


Figure 6.6: Geometry of the rolling tyre

The next numerical example considers the sound radiation of a rolling sphere. The motion of the source is a combination of rotation and translation:
While the center of the sphere moves with constant velocity along the x_1-axis, the source is rotating around an axis parallel to the x_2-axis through the center. This is shown in the picture above (see Figure 6.6).
Figure 6.7 shows the sound emitted by a source, which started at the front of the sphere and rolls over a microphone on the surface of the street at time 0.25s after $\frac{5}{4}$ rotations.

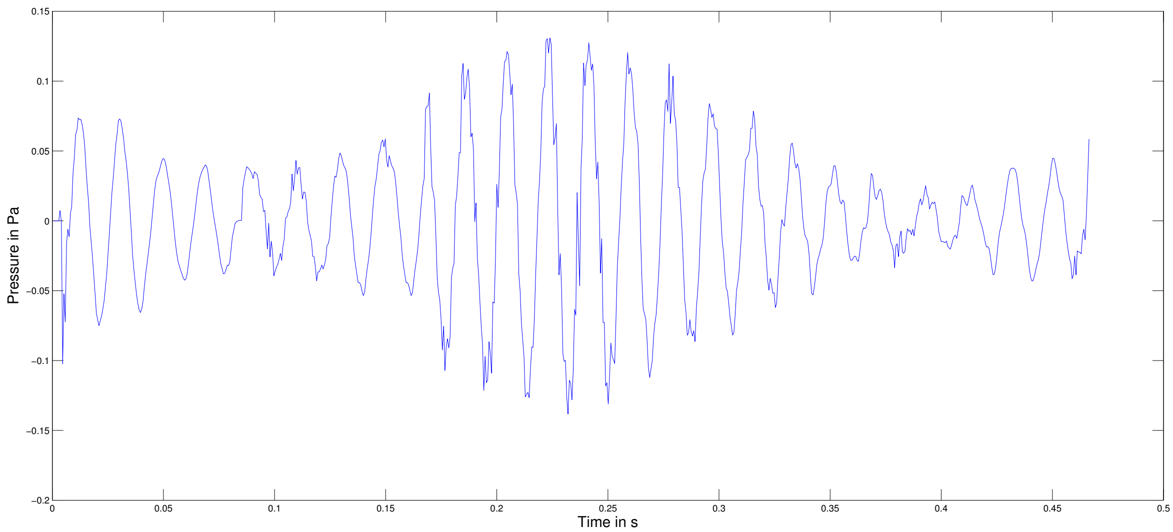


Figure 6.7: Sound pressure (in Pa) at an evaluation point 1.5 m away on the road for freq = 343 Hz.

6.4 Adaptive Methods for the Moving and Rolling Tyre

The a posteriori estimate in Theorem 4.3 continues to hold for the Dirichlet problem $V_R\dot{\phi} = \dot{f}$ for a moving or rolling tyre. The proof is identical, and only the constants of coercivity and continuity of the single layer potential depend on the velocity of the tyre.

While we could neglect the time derivative in the resulting indicator for the standing or moving tyre, this term becomes important once we rotate the data for a rolling tyre. Also, as the singularities might no longer stay in the same part of the tyre in this case, both space- and time-adaptive refinements are necessary as in Gläfke thesis [21].

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